GEOMETRIC QUANTIZATION VIA COTANGENT MODELS

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Abstract. We work out cotangent lift models for integrable systems with non-degenerate singularities which can be of elliptic, hyperbolic and focus-focus type. Those singularities naturally appear in polarizations on compact manifolds given by integrable systems. In particular any semitoric system (as the ones studied in [PVuN09, PVuN11]) gives rise to singularities of this type). These structures also show up in algebraic geometry naturally for instance in the K3 surface which can be viewed as a semitoric system. When it comes to considering their quantization: serveral models have been proposed but none of them can compete with the model of Kähler quantization (which cannot always be applied in this case) in terms of independence of the polarization and . We use different versions of the cotangent lift technique for different kind of singularities (in the sense of Williamson). We apply these models to define new geometric quantization for non-degenerate singularities. By complexifying these system, we obtain a unique cotangent model which allows us to unify. In contrast to the former works these models are finite dimensional on compact manifolds.

1. Introduction

Basic ideas on models

models that capture symmetry

Lagrangian symmetries

Regular case- Action-angle coordinates as cotangent models

Liouville one-forms as canonical choice of connection 1-form

Quantization as an attempt to tame quantum physics. Metaphysical problem: Can this be done? Several attempts: Geometric Quantization, Formalquantization, quantization by deformation depend on choices (polarization).

Can we find a universal model?

In this article bla bla we attempt to consider the cotangent lift as a way to quantize integrable systems. We do this for regular an nondegenerate singularities.

In particular we provide a unified approach to former attempts in the literature....

Do we have a universal model

Simplified models....imply simplified models in quantization

Date: November 2020.

Other applications: Applications in perturbation theory?

2. A crash course on geometric quantization

Kostant introduced the main ideas of geometric quantization in the 70s [Kos70] and, today, they remain useful and have applications in representation theory, a big variety of physical problems and many other fields. The quantization consists essentially in associating a Hilbert space $\mathcal{Q}$ to a symplectic manifold (M, ω) and, in geometric quantization, this Hilbert space is constructed using the sections of a complex line bundle $\mathbb{L}$. The model given by Kostant goes through the cohomology associated to the sheaf of flat sections of $\mathbb{L}$ and is well-adapted for real polarizations given by integrable systems and toric manifolds, which are symplectic manifolds endowed with an effective Hamiltonian action of a torus whose rank is half of the dimension of the manifold [Mir14a]. An important result of Delzant states the existence of a one-to-one correspondence between closed toric manifolds in dimension $2m$ and the *Delzant polytopes* on $\mathbb{R}^m$ [Del88]. The Delzant polytope gives the real geometric quantization of closed toric manifolds [Ham10] because, given a toric manifold, its real geometric quantization is completely determined by the count of integral points in the interior of its associated Delzant polytope.

Segal also proposed a way to quantize a Hamiltonian system consisting essentially in associating to the phase space a real Hilbert space $(\mathcal{F}, (,))$ of the states of one particle [Seg67], bringing a symplectic structure ω and a complex structure J such that the complexification H of $\mathcal{F}$ under J has a complex scalar product $(,)_{\mathcal{C}}$ defined as

$$(,)_{\mathcal{C}} = (,) + i\omega$$

and the Hamiltonian evolution of the system is expressed by a unitary flow. Although Segal quantization is quite useful for many purposes, we focus on Kostant's, since it is the geometric quantization that will be convenient to deal with singularities and using cotangent models.

Let (M, ω) be a symplectic manifold. A *prequantization line bundle* is a complex line bundle $\mathbb{L}$ over M, equipped with a connection ∇ whose curvature is ω. A *real polarization* is a foliation of M into Lagrangian submanifolds and we will usually want to compute the quantization of a compact and completely integrable system (M, ω, F) using the singular real polarization given by the singular foliation by levels of F, which are generically Lagrangian tori.

Definition 2.1. A section σ of $\mathbb{L}$ is *flat along the leaves* for *leafwise flat* if it is covariant constant along the fibres of F, with respect to the prequantization connection ∇. This means that $\nabla_X \sigma = 0$ for any vector field X tangent to fibres of F. Denote by $\mathcal{J}$ the sheaf of sections which are flat along the leaves.

Definition 2.2. With (M, ω, F), $\mathbb{L}$, and $\mathcal{J}$ as above, the *quantization* of M is

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M; \mathcal{J}).$$

Definition 2.3. A leaf $\mathfrak{l}$ of the (singular) foliation is a *Bohr-Sommerfeld leaf* if there is a leafwise flat section σ defined over all of $\mathfrak{l}$.

Although leafwise flat sections always exist locally (because by construction the curvature of ∇ is ω, which is zero when restricted to a leaf), we are requiring global existence, which is a strong condition. The set of Bohr-Sommerfeld leaves is discrete in the leaf space and a leaf is Bohr-Sommerfeld if and only if its holonomy is trivial around all the loops contained in the leaf.

The main result about quantization using real polarizations is a theorem of Śniatycki which states that, if the leaf space B^n is a manifold and the map $\pi\colon M^{2n} \to B^n$ is a fibration with compact fibres, then all of the cohomology groups are zero except in degree n [Ś77]. Furthermore, H^n can be expressed in terms of the Bohr-Sommerfeld leaves and the dimension of H^n is exactly the number of Bohr-Sommerfeld leaves.

In [HM10], Hamilton and Miranda prove the following key theorem for compact two-dimensional integrable systems with nondegenerate singularities.

Theorem 2.4. *Let (M,ω,F) be a two-dimensional, compact, completely integrable system, whose moment map has only nondegenerate singularities. Suppose M has a prequantum line bundle $\mathbb{L}$, and let $\mathcal{J}$ be the sheaf of sections of $\mathbb{L}$ flat along the leaves. The cohomology $H^1(M,\mathcal{J})$ has two contributions of the form $\mathbb{C}^{\mathbb{N}}$ for each hyperbolic singularity, each one corresponding to a space of Taylor series in one complex variable. It also has one $\mathbb{C}$ term for each non-singular Bohr-Sommerfeld leaf. That is,*

$$H^1(M;\mathcal{J}) \cong \bigoplus_{p\in\mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{b\in BS} \mathbb{C}_b. \tag{2.1}$$

The cohomology in other degrees is zero. Thus, the quantization of M is given by (2.1).

Two more important theorems were proved by Miranda, Presas and Solha in [MPS20], concerning the geometric quantization of focus-focus fibers.

Theorem 2.5. *The geometric quantization of a saturated neighborhood of a focus-focus fiber with n nodes is:*

- 0 *if the singular fiber is not Bohr–Sommerfeld.*
- *isomorphic to*
$$(C^\infty(\mathbb{R};\mathbb{C}))^{n_f},$$
if the singular fiber is Bohr–Sommerfeld, where $n_f = n$ (for compact fibers) and $n_f = n-1$ otherwise.

Theorem 2.6. *For a 4-dimensional closed almost toric manifold M, with n_r regular Bohr–Sommerfeld fibers and n_f focus-focus Bohr–Sommerfeld fibers:*

$$\mathcal{Q}(M) \cong \mathbb{C}^{n_r} \oplus \left(\bigoplus_{j\in\{1,\dots,n_f\}} (C^\infty(\mathbb{R};\mathbb{C}))^{n(j)} \right),$$

with $n(j)$ the number of nodes on the j-th focus-focus Bohr–Sommerfeld fiber.

These results allow to compute the quantization of some particular systems such as $K3$ surfaces (see Figure 1).

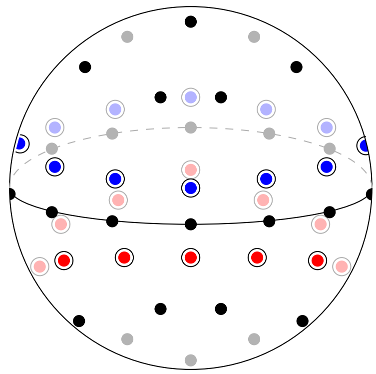

FIGURE 1. K3 surface as a singular fiber bundle over the sphere. Its quantization is isomorphic to $\mathbb{C}^{50}$. [MPS20]

Although the geometric quantization of the regular case is quite clear and for specific systems with nondegenerate singularities of hyperbolic and focus-focus type there do exist results, they are not so general.

3. Moment maps and Hamiltonian systems

Hamiltonian actions and moment maps are the absolute key concepts in the link between symplectic geometry and integrable systems. In this section we give a brief review on them, giving special attention to integrable systems with non-degenerate singularities.

Definition 3.1. An integrable system on M is given by a smooth map $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ such that $\{f_i, f_j\} = 0$ for all $1 \leq f_i, f_j \leq n$ and $\operatorname{rank} f = n$ almost everywhere. If the Hamiltonian vector fields provided by each f_i (i.e. the X_i satisfying $\iota_{X_i}\omega = -df_i$), which have commuting flows $\phi_t^1, \ldots, \phi_t^n$, are complete, then the system induces an $\mathbb{R}^n$ action on M, called the *joint flow*:

$$\rho : \mathbb{R}^n \times M \longrightarrow M \tag{3.1}$$

$$(t_1, \ldots, t_n, p) \longmapsto \phi_t^1 \circ \cdots \circ \phi_t^n(p) \tag{3.2}$$

Definition 3.2. Let $H \in \mathcal{C}^\infty$ be a smooth function on a symplectic manifold (M, ω) (in Physics, a Hamiltonian, the function of total energy). The *Hamiltonian vector field* X_H associated to H is defined as the only solution of $\iota_{X_H}\omega = -dH$.

Definition 3.3. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider also $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. Suppose $\psi : G \to \mathrm{Diff}(M)$ is an action on a symplectic manifold (M, ω). It is called a *Hamiltonian action* if there exists a map $\mu : M \to \mathfrak{g}^*$ which satisfies:

- For each $X \in \mathfrak{g}$, $d\mu^X = \iota_{X^\#}\omega$, i.e., μ^X is a Hamiltonian function for the vector field $X^\#$, where
  - $\mu^X : p \longmapsto \langle \mu(p), X \rangle : M \longrightarrow \mathbb{R}$ is the component of μ along X,
  - $X^\#$ is the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subset G$.
- The map μ is equivariant with respect to the given action ψ on M and the coadjoint action: $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$, for all $g \in G$.

Then, (M, ω, G, μ) is called a *Hamiltonian G-space* and μ is called the *moment map.*

The normal form of a completely integrable system around a whole leaf of a regular point is well-known by the Arnold-Liouville-Mineur theorem.

Theorem 3.4 (Arnold-Liouville-Mineur)**.** *Let (M^{2n}, ω) be a symplectic manifold. Let $f_1, \ldots, f_n$ functions on M which are functionally independent (i.e. $df_1 \wedge \cdots \wedge df_n \neq 0$) on a dense set and which are pairwise in involution. Assume that m is a regular point of $F = (f_1, \ldots, f_n)$ and that the level set of F through m, which we denote by $\mathcal{F}_m$, is compact and connected.*

Then, $\mathcal{F}_m$ is a torus and on a neighbourhood U of $\mathcal{F}_m$ there exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_n)$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_n)$ such that:

(1) The functions $(\theta_1, \ldots, \theta_n, p_1, \ldots, p_n)$ define a diffeomorphism $U \simeq \mathbb{T}^n \times B^n$.
(2) The symplectic structure can be written in terms of these coordinates as

$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$

(3) The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^n \times B^n$, in particular, the functions $f_1, \ldots, f_s$ depend only on $p_1, \ldots, p_n$.

The functions p_i are called action coordinates; the functions θ_i are called angle coordinates.

For singularities, and refined results have been obtained for local normal forms.

At singularities, one has to dig deeper because it can be very difficult to understand both the geometry and the dynamics of the system depending on the degeneracy of dF. For the case of non-degenerate singularities powerful results hav been obtained and, for instance, we have local normal forms.

Definition 3.5. A point $p \in M^{2n}$ is a *singular point* (or a *singularity*) of an integrable Hamiltonian system given by $F = (f_1, \ldots, f_n)$ if the rank of $dF = (df_1, \ldots, df_n)$ at p is less than n. The singular point p has *rank* k and *corank* of $n-k$ if $\mathrm{rank}(dF)_p = \mathrm{rank}\left((df_1)_p, \ldots, (df_n)_p\right) = k$.

Definition 3.6. Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing, i.e., if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then

$Y \in \mathfrak{h}$. If $\mathfrak{g}$ is finite-dimensional and semisimple over an algebraically closed field of characteristic zero, a Cartan subalgebra is a maximal abelian subalgebra (a subalgebra consisting of semisimple elements).

Definition 3.7. Let (M^{2n}, ω) be a symplectic manifold with an integrable Hamiltonian system of n independent and commuting first integrals $f_1, \ldots, f_n$. Consider a singular point $p \in M$ of rank 0, i.e. $(df_i)_p = 0$ for all i. It is called a *non-degenerate singular point* if the operators $\omega^{-1}d^2 f_1, \ldots, \omega^{-1}d^2 f_n$ form a Cartan subalgebra in the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{sp}(T_pM, \omega)$.

Remark 3.8. The operators $\omega^{-1}d^2 f_i$, where df_i is the Hessian of f_i, associate a function to the Hessian by visualizing the Hessian as a quadratic form $H(u, v)$ and taking the symplectic dual of the function obtained. A good reference for details of the algebraic construction of the Cartan subalgebra is [BF04].

The classification of non-degenerate critical points of the moment map in the real case was obtained by Williamson [Wil36]. In the complex case, all the Cartan subalgebras are conjugate and hence there is only one model for non-degenerate critical points of the moment map.

Theorem 3.9 (Williamson)**.** *For any Cartan subalgebra $\mathcal{C}$ of $\mathfrak{sp}(2n, \mathbb{R})$, there exists a symplectic system of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \ldots, f_n$ of $\mathcal{C}$ such that each of the quadratic polynomials f_i is one of the following:*

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 && \text{for } \ 1 \le i \le k_e \\
&f_i = x_i y_i && \text{for } \ k_e + 1 \le i \le k_e + k_h \\
&\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \end{cases} && \text{for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned}
$$

The three types are called elliptic, hyperbolic and focus-focus, respectively.

Remark 3.10. Notice that the focus-focus components always go by pairs. Because of theorem 3.9, the triple (k_e, k_h, k_f) at a singular point it is an invariant. It is also an invariant of the orbit of the integrable system through the point [Zun96].

If p is a non-degenerate singularity of the moment map F, it is characterized by four integer numbers, the rank k of the singularity and the triple (k_e, k_h, k_f). By the way they are defined, they satisfy $k + k_e + k_h + 2k_f = n$, where n is the number of degrees of freedom of the integrable system.

The following is a result of Eliasson [Eli90] and Miranda and Zung ([Mir03], [Mir14b], [MZ04]).

Theorem 3.11 (Smooth local linearization)**.** *Given an smooth integrable Hamiltonian system with n degrees of freedom on a symplectic manifold (M^{2n}, ω), the Liouville foliation in a neighborhood of a non-degenerate singular point of rank k and Williamson type (k_e, k_h, k_f) is locally symplectomorphic to the model Liouville* *foliation, which is the foliation defined by the basis functions of Theorem 3.9 plus "coordinate functions" $f_i = x_i$ for $i = k_e + k_h + 2j + 1$ to n.*

Remark 3.12. The theorem states the existence of a semilocal symplectomorphism between foliations with a non degenerate singularity of rank k and the same parameters (k_e, k_h, k_f). One could think that functions are also preserved via a symplectomorphism, but it is not possible to guarantee this statement when $h_k \neq 0$ as one can add up analytically flat terms on different connected components (see counterexample in [Mir03]). In general one needs more information about the topology of the leaf to conclude.

Remark 3.13. Because of Theorem 3.11, if one considers the Taylor expansions of $F = (f_1, \dots, f_n)$ at the non-degenerate singular point in a canonical coordinate system and removes all terms except for linear and quadratic, the functions obtained remain commuting and define a Liouville foliation that can be considered as the *linearization* of the initial foliation $\mathcal{F}$ given by $f_1, \dots, f_n$, to which it is symplectomorphic.

The description of non-degenerate singularities at the semilocal level is completed with the following two results.

Theorem 3.14 (Model in a covering)**.** *The manifold can be represented, locally at a non-degenerate singularity of rank k and Williamson type (k_e, k_h, k_f), as the direct product*

$$M^{reg} \times \overset{k}{\cdots} \times M^{reg} \times M^{ell} \times \overset{k_e}{\cdots} \times M^{ell} \times M^{hyp} \times \overset{k_h}{\cdots} \times M^{hyp} \times M^{foc} \times \overset{k_f}{\cdots} \times M^{foc}$$

Where:

- *M^{reg} is a "regular block", given by*
$$f = x,$$
- *M^{ell} is an "elliptic block", representing the elliptic singularity given by*
$$f = x^2 + y^2,$$
- *M^{hyp} is an "hyperbolic block", representing the hyperbolic singularity given by*
$$f = xy,$$
- *M^{foc} is a "focus-focus block", representing the focus-focus singularity given by*
$$\begin{cases} f_1 = x_1 y_2 - x_2 y_1 \\ f_2 = x_1 y_1 + x_2 y_2 \end{cases}.$$

For the first three types of blocks the symplectic form is $\omega = dx \wedge dy$, while for the focus-focus block it is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

In the case of a smooth system (defined by a smooth moment map), a similar result was proved and described by Miranda and Zung in [MZ04]. It summarizes some previously results proved independently and fixes the case where there are hyperbolic components ($k_h \neq 0$), because in this case the result is slightly different and it has to be taken the semidirect product in the decomposition. As opposite to the case where there are only elliptic and focus-focus singularities, in which the base of the fibration of the neighbourhood is an open disk, if there are hyperbolic components the topology of the fiber can become complicated. The reason is essentially that for the smooth case a level set of the form $\{x_i y_i = \varepsilon\}$ is not connected but consists of two components.

Theorem 3.15 (Miranda-Zung)**.** *Let $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ with coordinates $(p_1, ..., p_k)$ for D^k, $(q_1(mod 1), ..., q_k(mod 1))$ for $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ for $D^{2(n-k)}$ be a symplectic manifold with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$. Let F be the moment map corresponding to a singularity of rank k with Williamson type (k_e, k_h, k_f). There exists a finite group Γ, a linear system on the symplectic manifold V/Γ and a smooth Lagrangian-fibration-preserving symplectomorphism ϕ from a neighborhood of O into V/Γ, which sends O to the torus $\{p_i = x_i = y_i = 0\}$. The smooth symplectomorphism ϕ can be chosen so that via ϕ, the system-preserving action of a compact group G near O becomes a linear system-preserving action of G on V/Γ. If the moment map F is real analytic and the action of G near O is analytic, then the symplectomorphism ϕ can also be chosen to be real analytic. If the system depends smoothly (resp., analytically) on a local parameter (i.e. we have a local family of systems), then ϕ can also be chosen to depend smoothly (resp., analytically) on that parameter.*

Theorem 3.16. *Miranda* [Mir03] *Let ω be a symplectic form defined in a neighbourhood of the singularity at p for which the foliation $\mathcal{F}$ is Lagrangian. Then, there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, where x_i, y_i are local coordinates on $(\phi(U), p)$.*

4. The cotangent lift

The cotangent bundle of a smooth manifold can be naturally equipped with a symplectic structure in the following way. Let M be a differential manifold and consider its cotangent bundle T^*M. There is an intrinsic canonical linear form λ on T^*M defined pointwise by

$$\langle \lambda_p, v \rangle = \langle p, d\pi_p v \rangle, \qquad p = (m, \xi) \in T^*M, v \in T_p(T^*M),$$

where $d\pi_p : T_p(T^*M) \longrightarrow T_m M$ is the differential of the canonical projection at p. In local coordinates (q_i, p_i), the form is written as $\lambda = \sum_i p_i \, dq_i$ and is called the *Liouville 1-form*. Its differential $\omega = d\lambda = \sum_i dp_i \wedge dq_i$ is a symplectic form on T^*M.

Definition 4.1. Let $\rho : G \times M \longrightarrow M$ be a group action of a Lie group G on a smooth manifold M. For each $g \in G$, there is an induced diffeomorphism $\rho_g : M \longrightarrow M$. The *cotangent lift of ρ_g*, denoted by $\hat{\rho}_g$, is the diffeomorphism on

T^*M given by

$$\hat{\rho}_g(q,p) := (\rho_g(q), ((d\rho_g)_q^*)^{-1}(p)), \qquad (q,p) \in T^*M$$

which makes the following diagram commute:

$$\begin{tikzcd} T^*M \arrow[r, "\hat{\rho}_g"] \arrow[d, "\pi"] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, "\rho_g"] & M \end{tikzcd}$$

Given a difeomorphism $\rho : M \longrightarrow M$, its cotangent lift preserves the canonical form λ as the simple following computation shows. At a point $p = (m, \xi) \in T^*M$:

$$\begin{aligned} \lambda_p &= (d\pi)_p^* \xi = \\ &= (d\pi)_p^* (d\rho)_m^* \left((d\rho)_m^*\right)^{-1} \xi = \\ &= (d(\rho \circ \pi))_p^* \left((d\rho)_m^*\right)^{-1} \xi = \\ &= (d(\pi \circ \hat{\rho}))_p^* \left((d\rho)_m^*\right)^{-1} \xi = \\ &= (d\hat{\rho})_p^* (d\pi)_{\hat{\rho}(p)}^* \left((d\rho)_m^*\right)^{-1} \xi = \\ &= (d\hat{\rho})_p^* \lambda_{\hat{\rho}(p)}, \end{aligned}$$

where we have used the definitions of the Liouville 1-form and the cotangent lift and the fact that $\rho \circ \pi = \pi \circ \hat{\rho}$. Then, the canonical 1-form is preserved by $\hat{\rho}$.

As a consequence:

$$\hat{\rho}^*(\omega) = \hat{\rho}^*(d\lambda) = d(\hat{\rho}^*\lambda) = d\lambda = \omega.$$

So, the cotangent lift $\hat{\rho}_g$ preserves the Liouville form and the symplectic form of T^*M and we conclude the following standard result in the theory of cotangent lifts:

Example 4.2. Let $\rho : (\mathbb{R}^3, +) \times \mathbb{R}^3 \to \mathbb{R}^3$ be the Lie group action corresponding to a space translation defined by $\rho_x(q) = q + x$. Write (q,p) for an element of the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$.

By definition, $\hat{\rho}_x$, the cotangent lift of ρ_x is

$$\begin{aligned} \hat{\rho}_x(q,p) &= (\rho_x(q), ((d\rho_x)_q^*)^{-1}(p)) = \\ &= (q + x, ((Id^*)^{-1}(p)) = (q + x, p) \end{aligned}$$

Example 4.3. Let $\rho : SO(3, \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Lie group action defined by $\rho_A(q) = Aq$. Write (q,p) for an element of $T_q^*\mathbb{R}^3$. By definition, $\hat{\rho}_A$, the cotangent lift of ρ_A is

$$\hat{\rho}_A(q,p) = (\rho_A(q), ((d\rho_A)_q^*)^{-1}(p)) = (Aq, ((A^*)^{-1}(p)) = (Aq, Ap),$$

where the last equality holds because A is orthogonal. Like any cotangent lift, since the induced action in the cotangent bundle is Hamiltonian, it has an associated momentum map which, in this case, corresponds to the classical quantity $q \wedge p$.

5. A Mathematical perspective. Non-degenerate singularities as cotangent lifts

Cotangent lifts arise naturally in physics problems, and the link between integrable systems and cotangent models is clear in view of the following Kiesenhofer and Miranda result [KM17], which restates Theorem 3.4 to reveal that at a semilocal level the regular leaves are equivalent to a completely toric cotangent lift model.

Theorem 5.1. *Let $F = (f_1, \dots, f_n)$ be an integrable system on a symplectic manifold (M, ω). Then, semilocally around a regular Liouville torus, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

In the classical models of the harmonic oscillator, the simple pendulum and the spherical pendulum one already finds the three different types of non-degenerate singularities in its lowest dimensional case. A simple elliptic singularity is appears in the harmonic oscillator, a simple hyperbolic singularity shows up in the simple pendulum and a simple focus-focus singularity arises in the spherical pendulum.

5.1. **The hyperbolic singularity as a cotangent lift.** Take coordinates (x, y) on $T^*\mathbb{R}$ such that the symplectic form is $\omega = dx \wedge dy$ and the moment map is $f = xy$.

The Hamiltonian vector field associated to f is

$$X = (-x, y). \tag{5.1}$$

Consider the action of $\mathbb{R}$ on $\mathbb{R}$ given by:

$$\begin{array}{rccc} \rho: & \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t,x) & \longmapsto & e^{-t}x \end{array}.$$

Then, $((d\rho_t)^*_x)^{-1}$ acts as $y \longmapsto e^t y$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates (x, y) of $T^*\mathbb{R}$ is exactly:

$$\begin{array}{rccc} \hat{\rho}: & T^*\mathbb{R} & \longrightarrow & T^*\mathbb{R} \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}x \\ e^t y \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t = 0$, we obtain exactly $X = (-x, y)$, the vector field associated to the hyperbolic singularity.

5.2. **The elliptic singularity as a cotangent lift.** The cotangent lift in the elliptic case uses a complex moment map which is not holomorphic. It is a formal development and by no means holomorphicity is assumed.

Take complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$ such that the symplectic form is $\omega = \frac{i}{2} dz \wedge d\bar{z}$. The moment map corresponding to the elliptic singularity is $f = \frac{1}{2}\left(x^2 + y^2\right) = \frac{1}{2} z\bar{z}$.

The Hamilton's equations in this complex setting are:

$$\begin{gathered}
\iota_X \omega = -df \iff \\
\iff \iota_{\left(\alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial \bar{z}}\right)} \left(\frac{i}{2} dz \wedge d\bar{z}\right) = -\frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial \bar{z}} d\bar{z} \iff \\
\iff \frac{i\alpha}{2} d\bar{z} - \frac{i\beta}{2} dz = -\frac{1}{2} \bar{z} dz - \frac{1}{2} z d\bar{z} \iff \\
\iff \begin{cases} \alpha = iz \\ \beta = -i\bar{z} \end{cases}
\end{gathered}$$

Then, the Hamiltonian vector field associated to f is

$$X = (iz, -i\bar{z}). \tag{5.2}$$

Now, consider the following action of $\mathbb{R}$ on $\mathbb{C}$, which corresponds to a rotation of z of angle t:

$$\begin{array}{rccc} \rho: & \mathbb{R} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (t, z) & \longmapsto & e^{it} z \end{array}.$$

Then, $((d\rho_t)^*_z)^{-1}$ acts as $\bar{z} \longmapsto e^{-it}\bar{z}$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates $(z, \bar{z})$ of $T^*\mathbb{C}$ is:

$$\begin{array}{rccc} \hat{\rho}: & T^*\mathbb{C} & \longrightarrow & T^*\mathbb{C} \\ & \begin{pmatrix} z \\ \bar{z} \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{it} z \\ e^{-it} \bar{z} \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t = 0$ we obtain $X = (iz, -i\bar{z})$, the vector field associated to the elliptic singularity.

5.3. **The focus-focus singularity as a cotangent lift.** In a singularity of focus-focus type in a manifold of dimension 4, we can take coordinates (x_1, x_2, y_1, y_2) in a way that the symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the moment map is $F = (f_1, f_2) = (x_1 y_2 - x_2 y_1, x_1 y_1 + x_2 y_2)$.

The Hamiltonian vector fields associated to f_1 and f_2 are

$$X_1 = (x_2, -x_1, y_2, -y_1), \qquad X_2 = (-x_1, -x_2, y_1, y_2).$$

Let $G = S^1 \times \mathbb{R}$ and $M = \mathbb{R}^2$. Consider the action of a rotation and a radial dilation of $\mathbb{R}^2$ given by

$$\begin{aligned}\rho\colon (S^1 \times \mathbb{R}) \times \mathbb{R}^2 &\to \mathbb{R}^2 \\ ((\theta, t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) &\mapsto \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\end{aligned}$$

Then, the cotangent lift $\hat{\rho}$ associated to the group action is exactly

$$\hat{\rho}\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2 \tag{5.3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{-t}(x_1\cos\theta + x_2\sin\theta) \\ e^{-t}(-x_1\sin\theta + x_2\cos\theta) \\ e^{t}(y_1\cos\theta + y_2\sin\theta) \\ e^{t}(-y_1\sin\theta + y_2\cos\theta) \end{pmatrix} \tag{5.4}$$

Deriving the vector with respect to θ and evaluating at 0 we obtain exactly $X_1 = (x_2, -x_1, y_2, -y_1)$. While deriving the vector with respect to t and evaluating at 0 we obtain exactly $X_2 = (-x_1, -x_2, y_1, y_2)$.

5.4. **The focus-focus singularity as a complexification.** On another level, the focus-focus singularity can be seen as a complexification, that is, its moment map can be locally given by two complex coordinates. Take $F = (f_1, f_2) = (x_1y_2 - x_2y_1, x_1y_1 + x_2y_2)$, the moment map associated to the focus-focus singularity in symplectic coordinates (x_1, x_2, y_1, y_2) and define $f = f_1 + if_2$, a complex moment map. Define two complex variables z_1 and z_2 the following way:

$$z_1 = x_1 + ix_2, \quad z_2 = y_1 + iy_2.$$

In these new coordinates (z_1, z_2), in which $\omega = \Re(d\bar{z}_1 \wedge dz_2)$ (and $\lambda = \Re(d\bar{z}_1 \wedge dz_2)$), we have $f = \bar{z}_1 z_2$, and we can see that this model resembles the moment map of the hyperbolic singularity. In the same way that, in coordinates (x, y), we can see the hyperbolic singularity either given as the product xy or given as the difference of squares $s^2 - t^2$ via the definition of $s = (x+y)/2, t = (x-y)/2$, we can see f either as given as the product $\bar{z}_1 z_2$ or given as a sum of squares if we apply the following complex linear change of variables

$$\zeta_1 = \frac{\bar{z}_1 + z_2}{2}, \quad \zeta_2 = \frac{\bar{z}_1 - z_2}{2i}.$$

In the new coordinates ζ_1, ζ_2, in which $\omega = \Re(2id\zeta_2 \wedge d\zeta_1)$ (and $\lambda = \Re(2i\zeta_2 \wedge d\zeta_1)$), we have $f = \zeta_1^2 + \zeta_2^2$, which is really convenient to consider the quantization of the focus-focus singularity.

The conclusion we derive is that the focus-focus singularity can be seen as the cotangent lift of a non-compact action and as a complexification, but both approaches can not be made compatible with each other. We would say that the cotangent lift and the complexification do not commute. In fact, if it was possible to do it, then the focus-focus singularity as seen in Theorem 3.14 would not be a 4-dimensional elementary block but the product of two 2-dimensional elementary blocks, which is not the case (by Theorem 3.9, which proves that the focus-focus singularity is elementary).

6. Quantization via complexification and cotangent models

7. Complexification of a Lie group action

We have the following definitions of the complexification of a compact Lie group and a complexification of a Lie algebra.

Definition 7.1. Let K be a compact Lie group. An *analytic complexification* of K is a complex analytic group G together with a Lie group homomorphism $i : K \longrightarrow G$ such that, if $f : K \longrightarrow H$ is another Lie group homomorphism into a complex analytic group H, then there exists a unique analytic homomorphism $F : G \longrightarrow H$ such that $f = F \circ i$.

Definition 7.2. The complexification $\mathfrak{g}^{\mathbb{C}}$ of a Lie algebra $\mathfrak{g}$ is the space of pairs (X_1, X_2) of elements of $\mathfrak{g}$ with product by $a + ib \in \mathbb{C}$ given by

$$(a + ib)(X_1, X_2) = (aX_1 - bX_2, aX_2 + bX_1).$$

This definition makes it possible to think of the complexification of $\mathfrak{g}$ as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ extends in a natural way to a Lie bracket on $\mathfrak{g}^{\mathbb{C}}$ by:

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] - [Y_1, Y_2], [X_1, Y_2] + [Y_1, X_2]),$$

which can be thought as the following computation:

$$[X_1 + iY_1, X_2 + iY_2] = [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2])$$

In the following example we can see

Example 7.3. The Lie groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ have the same Lie algebra, since $SO(n, \mathbb{R})$ is the connected component of $O(n, \mathbb{R})$ that contains the identity. The complexification of the Lie algebra $\mathfrak{so}(n, \mathbb{R})$ of the real anti-symmetric matrices is naturally the Lie algebra of the complex anti-symmetric matrices $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$, since $\mathfrak{so}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{so}(n, \mathbb{R}) + i\mathfrak{so}(n, \mathbb{R}) = \mathfrak{so}(n, \mathbb{C})$.

The topology of the simple orthogonal group over the complex numbers is quite simple. As well as $SO(n, \mathbb{R})$, $SO(n, \mathbb{C})$ is a connected Lie group, since any element can be joined by a path to the identity. In the particular case of $n = 2$, the elements of $SO(n, \mathbb{C})$ can be thought as rotations and can be identified in a hyperbolic basis with the invertible elements of $\mathbb{C}$, i.e., with $\mathbb{C} \setminus \{0\}$. The topology of this set can be, at its turn, identified to the Cartesian product $S^1 \times \mathbb{R}$.

For our purposes it is only necessary to consider the complexification of a compact Lie group, which is quite more explicit do define. Suppose G is a closed subgroup of the unitary group $U(V)$ where V is a finite-dimensional complex inner product space. The complexification $G_{\mathbb{C}}$ of G consists of all operators g in $GL(V)$

such that $g\otimes^N$ commutes with $\mathrm{End}_G(V \oplus \mathbb{C})\otimes^N$ and g acts trivially on the second summand in $\mathbb{C}$. It is compatible with the the polar decomposition $g = u \cdot \exp iX$, where u is a unitary operator in the compact group and X is a skew-adjoint operator in its Lie algebra, and with the Cartan decomposition

$$G_{\mathbb{C}} = G \cdot P = G \cdot \exp i\mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of G and the exponential factor P is a positive operator which is invariant under conjugation by G.

7.1. **The focus-focus singularity as a complexified model.** We have just seen that the complexification of $S^1 \cong SO(2,\mathbb{R})$ gives $SO(2,\mathbb{C}) \cong S^1 \times \mathbb{R}$. On the other hand, in Section 5.3 we showed how the focus-focus singularity can be presented as a cotangent model, a model which precisely starts from the cotangent lift of the action of the non-compact Lie group $S^1 \times \mathbb{R}$. Our discussion at the end of that section, though, already made it clear that it is not possible to make the cotangent lift compatible with the complexification. If it was possible to find an action of a compact group (like S^1) which, after a cotangent lift and a complexification, provided the infinitesimal generator of the singularity, one would be able to apply ideas of rigidity used in [MM20] to the focus-focus singularity. But it is not expected to find any action of this type because the focus focus singularity can not be put as a product of two simpler singularities.

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