The b-geometry of magnetic fields

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Abstract

Among Poisson manifolds, b-symplectic manifolds constitute an exciting class, larger than symplectic manifolds, but still constrained enough to get crucial results. In this paper, we build bridges between these mathematical models and physical problems. We show that usual b-symplectic forms - canonical or twisted - can be interpreted physically. We also show that magnetism mixes well with b-geometry and give an original model of Hamiltonian dissipation.

1 Introduction

1.1 Background

Starting from physics, symplectic geometry is now one of the major subjects of mathematics ([Can01]). In a manifold, a 2-form ω is called symplectic if it is closed and non-degenerate. Closeness states that $d\omega = 0$, where d is the exterior derivative. Non-degeneracy states that for any vector field v, $i_v\omega$, where i is the interior product, is point-wisely an isomorphism when v does not vanish. A manifold with such a structure is called a symplectic manifold. Considering a function H over the manifold, called Hamiltonian, it is useful to consider its associated Hamiltonian flow, which is the flow of the vector field X defined by the equation $i_X\omega = dH$, where d is again the exterior derivative. Such a structure is called a Hamiltonian manifold. Let us remark that the existence and uniqueness of X comes for the non-degeneracy of the symplectic form.

One of the main theorems of mathematical physics appears naturally in this formalism, this is Noether's theorem ([Noe18]). This result starts by considering structure-preserving actions. One may study actions of any Lie group, but the simplest case is a 1-parameter group. The action is then generated by a vector field, let us say u, called the infinitesimal generator of the action. In such a situation, Noether's theorem gives the existence of a function over the manifold, let us say F, preserved by the action and conjugated to u by the symplectic form. Namely, $i_u\omega = dF$, where i is the interior product. That is, Noether's theorem gives a sufficient condition to be a Hamiltonian flow. This function F is called the momentum map. Moreover, in the case of a Hamiltonian manifold, a direct corollary is that this generator F is preserved by the Hamiltonian flow, which is an basic result of dynamics.

From a physical point of view, this result can be interpreted as very fundamental. Indeed, symplectic geometry is the natural language of mechanics ([LL60, JM99]). Starting from the configuration space, its cotangent bundle, called the phase space, is canonically symplectic. Calling q_i the space coordinates, coming from the configuration space, and p_i the momentum coordinates, coming form the cotangent bundle, the canonical symplectic form writes $\omega = \sum_i dp_i \wedge dq_i$ where $\wedge$ is the exterior product. Considering the physical Hamiltonian $\mathcal{H} = \sum_i \frac{1}{2}p_i^2 + V$ where the potential V only depends on space, the associated Hamiltonian flow describes exactly the physical dynamics and gives the usual Newton's laws.

Thanks to this interpretation, a structure preserving action corresponds to a physical symmetry. The associated momentum map F is now a physical quantity preserved by the dynamics and meaning physically its conservation through the time. For instance, the symmetry by translation gives the conservation of momentum, while the symmetry by rotation gives the conservation of the angular momentum. One major source of symmetries is the arbitrariness of the observer and then the choice of coordinates. For instance, the two previous examples come from the choice of the frame, that is the choice of the observer. Physically, this principle is crucial to understand the physics behind the calculations. Moreover, reduction also interests physicists since it can simplify the dynamical studies and solve many models.

In quantum physics, these principles also appear naturally ([CTDL18]). Indeed, the formalism of commutators is related to symplectic geometry, and Noether's theorem also gives important results. In dynamics, the symplectic formalism can be completed to include dissipation, this is the metriplectic formalism ([Mor84]). In addition to the Poisson bracket, coming from the symplectic formalism, there is a dissipative bracket, coming from out-of-equilibrium thermodynamics ([CM20]).

In physics, the usual formalism of dynamics is Poisson geometry ([JM99]). The Poisson manifolds are generalisations of symplectic manifolds. The symplectic form is then replaced by its inverse, a bivector that we denote $\mathcal{J}$. Indeed, ω may be seen as a smooth map from the space of vector fields $\chi(\mathcal{P})$ to the space of 1-forms $\Omega^1(\mathcal{P})$, where $\mathcal{P}$ is the name of the manifold. When we consider ω from this points of view, we will denote it $\omega^\flat$ to be clear. Non-degeneracy means that this map an isomorphism, its inverse being $\mathcal{J}^\flat$. Similarly, $\mathcal{J}$ may be seen as a bivector field or as a smooth map from the space of 1-forms $\Omega^1(\mathcal{P})$ to the space of vector fields $\chi(\mathcal{P})$, in the latter case, we denote it $\mathcal{J}^\flat$. To get a general Poisson manifold, the idea is not to require $\mathcal{J}^\flat$ to be invertible. Consequently, ω is no longer defined and we cannot use symplectic tools any more. A formal definition of Poisson structures is the following:

Definition 1 (Poisson Manifold) *A Poisson manifold $(\mathcal{P}, \mathcal{J})$ is a differential manifold $\mathcal{P}$ equipped with a smooth bivector field $\mathcal{J} \in \Gamma\left(\mathrm{T}^2\mathcal{P}\right)$, where Γ stands for the set of smooth sections, which fulfils the following properties:*

(i) $\mathcal{J}$ is skew-symmetric, that is, $\mathcal{J}(\alpha,\beta) = -\mathcal{J}(\beta,\alpha)$ for any 1-forms α, β.

(ii) $\mathcal{J}$ fulfils the Jacobi identity.

Among this large class of manifold, some of them are special. They are the b-symplectic manifolds, where the singularity is restricted to a hypersurface ([GMP14]). These manifolds are almost symplectic and many results from symplectic geometry remain true.

1.2 Basics of b-Geometry

Definition 2 (b-Manifold) *A b-manifold $(\mathcal{P}, Z)$ is a differential manifold $\mathcal{P}$ equipped with a hypersurface $Z \subset \mathcal{P}$. A b-vector field over $(\mathcal{P}, Z)$ is a vector field tangent to Z when restricted to Z. This definition allows us to consider the b-tangent bundle of $(\mathcal{P}, Z)$.*

This geometric definition may be written in coordinates. If z is a coordinate defining the critical set Z, and (x_i) the other coordinates, then a b-vector field is locally of the form $\sum_i f_i \partial_{x_i} + zg\partial_z$ where (f_i) and g are smooth functions. From this, one may define b-forms.

Definition 3 (b-Forms) *The b-cotangent bundle is the dual of the b-tangent bundle. Considering tensor products of this space, one defines b-forms.*

Locally, a b-form is $\alpha \wedge \frac{dz}{z} + \beta$ where α and β are smooth forms. The exterior derivative remains well-defined.

Definition 4 (*b*-Symplectic manifold) *The b-symplectic manifold* $(\mathcal{P}, Z, \omega)$ *is a b-manifold* $(\mathcal{P}, Z)$ *equipped with a closed b-2-form* ω *which is nondegenerate, meaning that* ω^n *is a non-vanishing b-volume form where* $2n$ *is the dimension of* $\mathcal{P}$.

Locally, a b-symplectic form is $dt \wedge \frac{dz}{z} + \sum dx_i \wedge dy_i$ where (x_i, y_i, t, z) is a set of coordinates.

2 Electromagnetism

2.1 Geometric construction

Considering the motion of a particle with mass m and charge e in an electromagnetic field, the configuration space Q is $\mathbb{R}^3$, with an electric potential function ϕ and a magnetic vector potential $\mathbf{A}$. The electric field is $\mathbf{E} = \nabla\phi$ and the magnetic field is $\mathbf{B} = \nabla \times \mathbf{A}$. In Newton's equation of motion $\mathbf{F} = m\mathbf{a}$, $\mathbf{F}$ is the Lorentz force $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, a function of both position and velocity.

In the Hamiltonian formalism, $\mathbf{A}$ becomes a 1-form by the identification between tangent and cotangent vectors induced by a fixed Riemannian metric. This identification takes $\mathbf{A}$ to the 1-form $< \mathbf{A}, \cdot >$. Then, $\mathbf{B}$ is the 2-form $d\mathbf{A}$. The configuration space Q can be any manifold (not necessarily of dimension 3). The magnetic field is a 2-form $\mathbf{B}$ on Q. The Maxwell equation $\nabla \cdot \mathbf{B} = 0$ becomes the condition $d\mathbf{B} = 0$. We define a new phase space $T^*_{\mathbf{B}}Q = (T^*Q, \omega_{\mathbf{B}})$ to be T^*Q with the symplectic form $\omega_{\mathbf{B}}$ which is the sum of the canonical form ω_Q and the pullback of $\mathbf{B}$ by the natural projection $T^*Q \to Q$, i.e., $\omega_{\mathbf{B}} = \omega_Q + \pi^*\mathbf{B}$. The charge e is set to 1. If $\mathbf{A}$ is any 1-form on Q, the "gauge transformation" given by translation by $\mathbf{A}$ takes $\omega_{\mathbf{B}}$ to $\omega_{\mathbf{B}-d\mathbf{A}}$. In particular, $T^*_{\mathbf{B}}Q$ is symplectomorphic to T^*Q if and only if $\mathbf{B}$ is exact. We assume that Q is provided with a Riemannian metric, which implicitly includes the mass, so that the metric identifies velocity $\mathbf{v}$ in TQ with momentum in T^*Q, by the same way that it identifies the potential vector with the 1-form $\mathbf{A}$. By abuse of notation, we will also denote the momentum itself by $\mathbf{v}$. The Hamiltonian for the Lorentz flow is then $\frac{1}{2}||\mathbf{v}||^2 + \phi$, with the magnetic field encoded in the symplectic form $\omega_{\mathbf{B}}$.

2.2 Nature of the magnetic form

To get a symplectic form $\omega_{\mathbf{B}}$, we need $\mathbf{B}$ to be exact. This digression discuss if it is common. First, can $\mathbf{B}$ be non-closed? Second, can it be closed but not exact? We answer yes to these two questions and provide examples for these three situations.

Setup: we work in the torus T^3 and denote (θ, ϕ, ψ) the coordinates. The volume form is, as usual, $\Omega = d\theta \wedge d\phi \wedge d\psi$. We choose the circles to have

2π as perimeter. Let us consider a magnetic vector field $\mathbf{B} = f(\theta, \psi)\partial_\psi$. The magnetic term becomes $\mathbf{B} = f(\theta, \psi)d\theta \wedge d\phi$.

B is not closed. We define $f(\psi) = sin(\psi)$ which is globally defined. That is, $\mathbf{B} = sin(\psi)d\theta \wedge d\phi$ and $d\mathbf{B} = cos(\psi)d\theta \wedge d\phi \wedge d\psi = cos(\psi)\Omega$. Then, $\mathbf{B}$ is not closed.

B is closed but not exact. We define $f = 1$. That is, $\mathbf{B} = d\theta \wedge d\phi$. Let us imagine that $\mathbf{B}$ is exact, that is $\mathbf{B} = d\alpha$. We write, without loss of generality, $\alpha = Fd\theta + Gd\phi$ with F and G depending on θ and ϕ. The equation $\mathbf{B} = d\alpha$ writes $\partial_\theta G - \partial_\phi F = 1$. Now G is periodic in θ and then $\partial_\theta G$ is periodic with zero mean value. Then, $\int \partial_\theta G\Omega = \int d\phi d\psi \int \partial_\theta G d\theta = 0$. Similarly, $\int \partial_\phi F\Omega = 0$. Integrating our equation then gives $0 = (2\pi)^3$, which is a contradiction.

B is exact. We define $f(\theta) = cos(\theta)$. That is, $\mathbf{B} = cos(\theta)d\theta \wedge d\phi$. We have $B = d(sin(\theta)d\phi)$, so $\mathbf{B}$ is exact.

To conclude this paragraph, the magnetic term in the complete symplectic form could destruct the closeness of the form. Then, it would no longer be a symplectic form and Jacobi identity would no longer hold, which is physically a problem. That is, we need to impose constraints on the form of the magnetic field to get a physical dynamics, which corresponds to the divergence-free of the magnetic field. This assumption physically means that there is no magnetic monopoles ([HM20]).

2.3 A physical example

In this example, we illustrate another interest of closeness of the magnetic field, which is Noether's theorem. We also use this example to illustrate the use of b-geometry to study asymptotic behaviours.

Let us consider $M = \mathbf{R}^3$ and $T^*M = \mathbf{R}^6$. The usual volume form is $\Omega = rdr \wedge d\theta \wedge dz$. In cylindric coordinates, we consider $\mathbf{B} = \frac{1}{r}f(r, z)\partial_z$. The magnetic term is $\mathbf{B} = f(r, z)dr \wedge d\theta$. The symplectic form being $\omega = d\theta \wedge dp_\theta + dr \wedge dp_r + dz \wedge dp_z + f(r, z)dr \wedge d\theta$.

We consider the action of S^1 by rotation given by ∂_θ. In the coordinates, the cotangent leaves action is also generated by ∂_θ. This action is a symmetry, which is clear physically and can be formally written as

$$\mathcal{L}_{\partial_\theta}\omega = di_{\partial_\theta}(d\theta \wedge dp_\theta) + i_{\partial_\theta}(\partial_z f dr \wedge d\theta \wedge dz) - d(f dr) = 0.$$

We calculate $i_{\partial_\theta}\omega = dp_\theta - f(r, z)dr$. We want this form to be exact. First, it must be closed, that is $\partial_z f = 0$. This is natural since otherwise the symplectic form would not be closed... Then, the space being contractile, it is exact. If $F(r)$ is a primitive of $f(r)$, we see that $\mathbf{B} = d(F(r)d\theta)$ and $i_{\partial_\theta}\omega = d(p_\theta - F(r))$.

Finally, the momentum map is $\mu = p_\theta - F(r)$. Using the modified Liouville form $\lambda = p_\theta d\theta + p_r dr + p_z dz - F(r)d\theta$, which fulfils $d\lambda = \omega$, we have the usual formula $< \mu, \alpha >=< \lambda, \alpha\partial_\theta >$.

Now, let us change variables to get b-geometry. We define $\pi = e^{p_\theta}$, that is $dp_\theta = \frac{d\pi}{\pi}$. The symplectic form becomes a twisted b-symplectic form: $\omega = d\theta \wedge \frac{d\pi}{\pi} + dr \wedge dp_r + dz \wedge dp_z + f(r,z)dr \wedge d\theta$. The Liouville form changes too: $\lambda = ln(\pi)d\theta + p_r dr + p_z dz - F(r)d\theta$. Finally, the momentum map writes $< \mu, \alpha >=< \lambda, \alpha\partial_\theta >= \alpha(ln(\pi) - F(r))$.

Physically, this is a particle moving in a magnetic vector field depending only on r. The change of coordinates brings (minus) infinite angular velocity at zero. If we add a potential $V(r)$ going to $-\infty$ far away to the Hamiltonian, the potential will provide infinite energy and the magnetic field could make the particle rotating so that the angular velocity diverges for infinite time. The change of coordinates then brings this asymptotic behaviour to a finite one.

3 Some Physical Interpretations

3.1 Canonical and twisted b-cotangent lifts

The physical setting for symplectic geometry is to consider a configuration space and its cotangent bundle. The physical symplectic form is the derivative of the Liouville form $\lambda = \sum_i p_i dq_i$. In b-geometry, there are two natural choices for the Liouville form, giving two different symplectic forms.

1. Canonical type, $\lambda = \frac{p_1}{q_1} dq_1 + \sum_{i>1} p_i dq_i$ and ${}^b\omega^c = \frac{1}{q_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$
2. Twisted type, $\lambda = log(p_1)dq_1 + \sum_{i>1} p_i dq_i$ and ${}^b\omega^t = \frac{1}{p_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$

The canonical symplectic form assumes the discontinuity is at the base (the transversal hypersurface Z is given by $q_1 = 0$), while the twisted symplectic form assumes the discontinuity is at the fiber (the transversal hypersurface Z is given by $p_1 = 0$).

3.2 Canonical interpretation for the non-twisted case

Considering the canonical b-symplectic form, the physical interpretation is usual. It was indeed the first motivation to b-geometry and gave the name of b, coming from *boundary*. It consists in pushing the boundary to infinity to get the physical setting. Here, we write the same argument adding the magnetic field.

Let us work in $\mathbb{R}^3$ and start from the physical setting. We impose $X > 0$ and regularize the boundary by considering $x = ln(X)$. The canonical symplectic form then becomes $\omega = \frac{dX\wedge dp_x}{X} + dy \wedge dp_y + dz \wedge dp_z$. The canonical metric becomes $g = \frac{dX^2}{X^2} + dy^2 + dz^2$ with volume form $\frac{dX\wedge dy\wedge dz}{X}$.

Now we add the contribution of the magnetic vector field given by $\mathbf{B} = b(X, y, z, p_x, p_y, p_z)\partial_z$. The direction is generic since the ∂_X direction does not mix with the singularity. Since the volume form is $\frac{1}{X} dX \wedge dy \wedge dz$, the associated magnetic form is $\mathbf{B} = \frac{b(X,y,z,p_x,p_y,p_z)}{X} dX \wedge dy$. Then, the b-symplectic form is

$$\omega_{\mathbf{B}} = \frac{1}{X} dX \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z + \frac{b}{X} dX \wedge dy \tag{3.1}$$

The physical Hamiltonian is $H(\mathbf{x}, \mathbf{p}) = \frac{p_x^2+p_y^2+p_z^2}{2} + V(\mathbf{x})$. The equations of motion are obtained by the usual calculus

$$\iota_{f_1 \frac{\partial}{\partial X} + f_2 \frac{\partial}{\partial y} + \cdots + f_4 \frac{\partial}{\partial p_X} + f_5 \frac{\partial}{\partial p_y} + f_6 \frac{\partial}{\partial p_x}} \omega_B = -dH = \frac{\partial H}{\partial X} dX + \frac{\partial H}{\partial y} dy + \frac{\partial H}{\partial z} dz \cdots \tag{3.2}$$

From this, we can calculate the Hamilton's equations:

$$\begin{cases} \dot{X} = X p_x \\ \dot{y} = p_y \\ \dot{z} = p_z \\ \dot{p}_x = -b p_y - X \partial_X V \\ \dot{p}_y = b p_x - \partial_y V \\ \dot{p}_z = -\partial_z V \end{cases} \tag{3.3}$$

Making the change of coordinates $X \to x$, we get back the usual equations of motion. To conclude this paragraph, we have shown that sending the boundary to infinity transforms b-geometry into classical symplectic geometry. The associated b-metric and b-magnetic field also transform well and naturally. Finally, it makes a consistent framework.

3.3 Canonical interpretation for the twisted case

For the twisted case, a similar interpretation is possible by bringing infinite velocity to zero. Also, this is compatible with the use of magnetism. In this section, we will explicitly give the associated change of variable and prove that it is the only one giving the physical system under acceptable assumptions.

We consider the twisted b-symplectic form $\omega = \frac{1}{p_x} dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$ and the usual volume form $dx \wedge dy \wedge dz$. Indeed, the singularity being in impulsion, there cannot be a spatial singularity. Our magnetic field is, without loss of generality, $B = a\partial_x + b\partial_z$ where a and b are smooth functions. Then, the total magnetic symplectic form is

$$\omega_B = \frac{1}{p_x} dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z - a dy \wedge dz - b dx \wedge dy.$$

We choose the generic Hamiltonian $H = F(p_x)G(x) + \frac{1}{2}p_y^2 + \frac{1}{2}p_z^2 + f(\mathbf{q})$, whose form is natural. At this point, F and G are general function. Our goal is to choose them so we get back usual physics. Let us calculate Hamilton's equations in detail. First,

$$dH = F'(p_x)G(x)dp_x + F(x)G'(x)dx + p_y dp_y + p_z dp_z + \partial_x f dx + \partial_y f dy + \partial_z f dz.$$

Writting the Hamiltonian vector field as $X = \alpha_x\partial_x + \alpha_y\partial_y + \alpha_z\partial_z + \beta_x\partial_{p_x} + \beta_y\partial_{p_y} + \beta_z\partial_{p_z}$, we have

$$\iota_X\omega_B = \frac{\alpha_x}{p_x}dp_x + \alpha_y dp_y + \alpha_z dp_z + (\alpha_y b - \frac{\beta_x}{p_x})dx + (\alpha_z a - \beta_y - \alpha_x b)dy - (\alpha_y a + \beta_z)dz.$$

We now identify terms. Starting with the impulsion 1-forms, we get

$$\begin{cases} \alpha_x = p_x F'(p_x)G(x) \\ \alpha_y = p_y \\ \alpha_z = p_z \end{cases} \tag{3.4}$$

and then

$$\begin{cases} \beta_x = -p_x F(p_x)G'(x) + bp_x p_y - p_x\partial_x f \\ \beta_y = -bp_x F'(p_x)G(x) + ap_z - \partial_y f \\ \beta_z = -ap_y - \partial_z f \end{cases} \tag{3.5}$$

That is, the equations of movement are:

$$\begin{cases} \dot{x} = p_x F'(p_x)G(x) \\ \dot{y} = p_y \\ \dot{z} = p_z \\ \dot{p}_x = -p_x F(p_x)G'(x) + bp_x p_y - p_x\partial_x f \\ \dot{p}_y = -bp_x F'(p_x)G(x) + ap_z - \partial_y f \\ \dot{p}_z = -ap_y - \partial_z f \end{cases} \tag{3.6}$$

We want to change variables $\tilde{x} = \varphi(x)$ and to introduce its impulsion $\tilde{p}_x$ so that: first,

$$\dot{\tilde{x}} = p_x F'(p_x)\varphi'(x)G(x) = \tilde{p}_x;$$

second, the Hamiltonian becomes

$$H = F(p_x)G(x) = \frac{1}{2}\tilde{p}_x^2 = \frac{1}{2}p_x^2 F'(p_x)^2\varphi'(x)^2G(x)^2.$$

This gives two equations, up to a multiplicative constant which is, without loss of generality 1:

$$\begin{cases} p_x^2F'(p_x)^2 = 2F(p_x) \\ \varphi'(x)^2G(x) = 1 \end{cases} \tag{3.7}$$

meaning

$$\begin{cases} \frac{dF}{\sqrt{F}} = \sqrt{2}\frac{dp_x}{p_x} \\ G(x) = \frac{1}{\varphi'(x)^2}. \end{cases} \tag{3.8}$$

then, not considering constants,

$$\begin{cases} F(p_x) = \frac{1}{2}ln(p_x)^2 \\ G(x) = \frac{1}{\varphi'(x)^2}. \end{cases} \tag{3.9}$$

From this, H is not formally a b-function, but almost if G is constant, meaning that φ is an affine function. A translation corresponding only to the addition of an associated logarithm of p_x is the Hamiltonian, it is not interesting. That is, $\varphi(x) = \lambda x$. In conclusion, $\tilde{p_x} = \frac{1}{\lambda}ln(p_x)$ and $H = \frac{1}{2\lambda^2}ln(p_x)^2 = \frac{1}{2}\tilde{p}_x^2$.

Now, let us see what are the equations for the impulsion. Of course, p_z is not interesting.

$$\begin{cases} \dot{p}_x = bp_xp_y - p_x\partial_x f \\ \dot{p}_y = -\frac{b}{\lambda^2}ln(p_x) + ap_z - \partial_y f = -\frac{b}{\lambda}\tilde{p}_x + ap_z - \partial_y f \end{cases} \tag{3.10}$$

Let us rewrite the first equation:

$$\lambda\dot{\tilde{p}}_x = bp_y - \partial_x f \tag{3.11}$$

Up to a dilation, we may choose $\lambda = 1$. The equations become

$$\begin{cases} \dot{x} = \tilde{p}_x \\ \dot{y} = p_y \\ \dot{z} = p_z \\ \dot{\tilde{p}}_x = bp_y - \partial_x f \\ \dot{p}_y = -b\tilde{p}_x + ap_z - \partial_y f \\ \dot{p}_z = -ap_y - \partial_z f \end{cases} \tag{3.12}$$

They are the usual equations but generated with another Hamiltonian,

$$H = \frac{1}{2}ln(p_x)^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_z^2 + f(\mathbf{q})$$

and the twisted symplectic form. The correct and only change of variables to get the physical system is then, as expected, $p_x \to ln(p_x)$. Also, we see that the magnetic part does not mix with the singularity and then there is not a complication.

Such a change of variables is now allowed thank to b-geometry and may be useful to study some asymptotic behaviours. A basic example of this was discussed in subsection 2.3

3.4 An original model of Hamiltonian dissipation

We work in $M = \mathbb{R}$ and $T^*M \cong \mathbb{R}^2$, with the twisted b-symplectic setting, we use the Hamiltonian

$$H(q,p) = \frac{p^2}{2} + f(q) \tag{3.13}$$

The Hamilton's equations derived from $\iota_{X_H}{}^b\omega^t = -dH$ are:

$$\begin{cases} \dot{q} = p^2 \\ \dot{p} = -p\frac{\partial f}{\partial q} \end{cases} \tag{3.14}$$

We do not study the trivial situation, so initially $p > 0$. As long as this remains true, we may define $y = ln(p)$. Then, $\dot{y} = -\partial_q f$ and since $ln(\dot{q}) = 2ln(p) = 2y$, we find

$$\ddot{q} = -2\dot{q}\partial_q f.$$

The special case $f = \frac{\lambda}{2}q$ is exactly the situation of a free massive particle with viscous friction. This is exceptional because friction is non-conservative and then cannot be described by the usual basic Hamiltonian setup. Now, the critical surface means zero velocity. Indeed, viscous friction cannot bring you to this point in finite time. This model, using a general viscous integral f could bring complex behaviours. Now, it is possible to use Hamiltonian tools to study it, for instance for Hamiltonian simulations. The limit of this model is that it is unidimensional and that no additional force can be added. These general models seems, up to now, too viscous to work in such a setup.

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References

[Can01] Ana Cannas. *Lectures on Symplectic Geometry.* Springer-Verlag, 2001.

[CM20] Baptiste Coquinot and Philip J. Morrison. *A general metriplectic framework with application to dissipative extended magnetohydrodynamics*, volume 86 of *3.* Journal of Plasma Physics, 2020.

[CTDL18] Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloë. *Mécanique quantique.* EDP-Sciences, 2018.

[GMP14] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and poisson geometry on b-manifolds. *Advances in Mathematics*, 264:864 – 896, 2014.

[HM20] J. Heninger and P. J. Morrison. Hamiltonian nature of monopole dynamics. *Physics Letters A*, 384(126101), 2020.

[JM99] Tudor Ratiu J.E. Marsden. *Introduction to Mechanics and Symmetry*. Springer-Verlag, 1999.

[LL60] L. D. Landau and E. M. Lifshitz. *Course of Theoretical Physics: Mechanics*. Pergamon Press, 1960.

[Mor84] P. J. Morrison. Bracket formulation for irreversible classical fields. *Phys. Lett. A*, 100:423–427, 1984.

[Noe18] Emmy Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:235–237, 1918.