Rigidity of b-cotangent lifts

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We follow the recipe of [GGK02] and adapt it to b-manifolds and b-symplectic manifolds.

1 The b-symplectic version

It is a natural question to ask if the same results obtained for the standard cotangent lift in the symplectic setting are also true in the b-symplectic case. Since rigidity of the cotangent lift is proved using Palais Theorem, in order to prove rigidity in the b-symplectic setting one has to use a b version of Palais Theorem. At its turn, the proof of Palais Theorem uses the classical Mostow-Palais Theorem on equivariant embeddings, so one has too prove also its b version. In this section, we first state the b version of the Mostow-Palais and the Palais Theorems. Then, we give the proofs of the b-symplectic Palais and the twisted b-cotangent lifted Palais Theorems, which are all new results.

1.1 The b-Mostow-Palais and the b-Palais Theorems

We want to prove the b-versions of the Mostow-Palais embedding Theorem and the Palais Rigidity Theorem. We give a proof which is similar to the proof of Palais Rigidity Theorem that can be found on [GGK02].

We start proving the following Lemma.

Lemma 1.1. *Representation b-functions are $\mathcal{C}^1$-dense in $\mathcal{C}^\infty(M, Z)$.*

Proof. Representation b-functions on G, with respect to the left or right action of G on itself, are uniformly dense in $\mathcal{C}^1(G)$. This follows from the Peter-Weyl Theorem adapted for b-functions. Namely, the matrix coefficients of a finite-dimensional representation T of G span the finite-dimensional representation $T^* \otimes T$ in $\mathcal{C}^\infty(G)$. Then, any linear combination of matrix coefficients is a representation b-function on G. On the other hand, by same the Peter-Weyl Theorem, such linear combinations are uniformly dense in $\mathcal{C}^\infty(G)$.

To prove the lemma, we will show that the *convolution* of a function f on M and a representation function u on G, is a representation function on M. More specifically, for $u \in \mathcal{C}^\infty(G)$ and $f \in \mathcal{C}^\infty(M, Z)$, we set:

$$f_u(x) = \int_G u(g) f(g^{-1}x) dg \tag{1.1}$$

where dg is the normalized Haar measure on the compact group G. And now we see that f_u is a b-representation function whenever u is a b-representation function. Indeed, for every $h \in G$, we have

$$hf_u = f_{hu} \tag{1.2}$$

where $hu(g) := u(h^{-1}g)$. This can be checked through the following calculation:

$$(hf_u(x)) = f_u(h^{-1}x) = \int_G u(g) f(g^{-1}h^{-1}x) dg = \int_G u(h^{-1}g) f(g^{-1}x) dg = f_{hu}(x). \tag{1.3}$$

For a fixed f, the mapping $u \mapsto f_u$ is linear and G-equivariant (as $hf_u = f_{hu}$). Hence, $V(f_u)$ is the image of $V(u)$ and, as a consequence, $V(f_u)$ is finite-dimensional if $V(u)$ is finite-dimensional.

To finish the proof, observe that every function f can be arbitrarily close $\mathcal{C}^1$-approximated by $b$functions of the form f_v with $v \in \mathcal{C}^\infty(G)$. It suffices to take as v a bump function on G which is supported near $e \in G$ and has unit integral. Uniformly approximating v by representation functions u, we obtain a $\mathcal{C}^1$-approximation of f by representation b-functions f_u. □

Before stating the b-Mostow Palais Embedding Theorem, We recall that a G-action on M induces a linear G-representation on $\mathcal{C}^\infty(M, Z)$, where $g \in G$ acts by sending $f \in \mathcal{C}^\infty(M, Z)$ to the b-function $(gf)(x) = f(g^{-1}x)$. Denote by $V(f)$ the span in $\mathcal{C}^\infty(M, Z)$ of the orbit $G \cdot f$. f is said to be a *representation b-function* if $V(f)$ is finite-dimensional.

Theorem 1.2 (b-Mostow-Palais embedding Theorem)**.** *Let a compact Lie group G act on a compact b-manifold (M, Z) via b-maps. Then, there exists an equivariant embedding of (M, Z) into a linear representation of G on a finite-dimensional pair (V, H) of vector spaces where H has codimension 1 in V.*

Proof. The evaluation maps $\delta_x : f \mapsto f(x)$, for $x \in (M, Z)$, give rise to an equivariant injection $x \mapsto \delta_x$ of M, Z into the dual space to $\mathcal{C}^\infty(M, Z)$. For every $f \in \mathcal{C}^\infty(M, Z)$, the space $V(f)$ is naturally a G-representation, and the evaluation map gives rise to an equivariant mapping $(M, Z) \to V(f)^*$. By Whitney's

Theorem, every manifold can be smoothly embedded in $\mathbb{R}^m$ for some m, and a C^1 deformation of an embedding remains an embedding Then, since representation b-functions are C^1-dense in $\mathcal{C}^\infty(M,Z)$ (by Lemma 1.1), there exists an embedding $(M,Z) \to \mathbb{R}^m$ whose components $f_1,\dots,f_m$ are representation b-functions.

Therefore, we obtain an equivariant evaluation map from (M,Z) to the direct sum $V = V(f_1)^* \oplus \cdots \oplus V(f_m)^*$. This map is an embedding because its composition with a suitable linear mapping $V \to \mathbb{R}^m$ is the original embedding $(f_1,\dots,f_m)$. And if the G action on M is effective, so is the G action on V. Then, G embeds in $GL(N)$, where $N = \dim V$. The action becomes orthogonal by taking any inner product on V and averaging with respect to the G-action. □

Now, before proving the b-Palais Rigidity Theorem, we need the following Proposition.

Proposition 1.3. *Let G be a compact Lie group and V a finite-dimensional vector space. Let ρ_0 be a linear representation of G in V. Then, for every representation ρ which is sufficiently C^0-close to ρ_0, there exists an automorphism A of V which intertwines ρ_0 and ρ, i.e., such that $\rho_0 = A \circ \rho \circ A^{-1}$. Besides, A can be chosen to depend smoothly on ρ and ρ_0.*

Proof. For each representation ρ of G in V, define a linear map $A: V \to V$ by

$$A(x) = \int_G \rho_0(g^{-1})\rho(g)(x)dg, \quad x \in V.$$

Then, for any $h \in G$,

$$A(\rho(h)x) = \int_G \rho_0(g^{-1})\rho(g)\rho(h)(x)dg = \int_G \rho_0(g^{-1})\rho(gh)(x)dg.$$

The change of variable $g \mapsto gh^{-1}$ turns the integral into

$$\int_G \rho_0(hg^{-1})\rho(g)dg = \rho_0(h)\int_G \rho_0(g^{-1})\rho(g)dg = \rho_0(h)A(x).$$

This shows that $A \circ \rho = \rho_0 \circ A$. When A is invertible, the proof is finished. Because A depends continuously on ρ and is equal to the identity map when $\rho = \rho_0$, the map A is invertible if ρ is sufficiently close to ρ_0. □

Theorem 1.4 (b-Palais Theorem)**.** *Let ρ be a b-action of a compact group G on a compact b-manifold (M,Z). For every b-action ρ_1 of G on (M,Z) which is sufficiently C^1-close to ρ, there exists a diffeomorphism $\phi: (M,Z) \to (M,Z)$ which is a b-map and which conjugates the actions: $\rho_1 = \phi\rho\phi^{-1}$. Also, it belongs to the connected component of the identity map.*

Proof. We start applying Theorem 1.2 to both ρ and ρ_1. We obtain two representations of G, say $\bar{\rho}$ and $\bar{\rho}_1$, on vector spaces V and V_1, respectively, and equivariant embeddings $M \to V$ for ρ and $(M, Z) \to V_1$ for ρ_1. We can find a linear isomorphism $V_1 \to V$ after which the embeddings become C^1-close and $\bar{\rho}$ and $\bar{\rho}_1$ become close, making it possible to identify $V = V_1$.

By Proposition 1.3, there exists a linear mapping $A : V \to V$, close to the identity, which sends $\bar{\rho}_1$ to $\bar{\rho}$ ($\bar{\rho} = A \circ \bar{\rho}_1 \circ A^{-1}$). Thus, we assume that the representations are equal and the embedding ψ_1 for ρ_1 is still C^1-close to the embedding ψ for ρ.

The image of ψ_1 lies in a small tubular neighborhood of the image of ψ, which we identify with (M, Z). Let us fix a G-invariant metric on V. The composition ϕ of ψ_1 with the orthogonal projection from the tubular neighborhood to (M, Z) is clearly G-equivariant: $(M, Z, \rho_1) \to (M, Z, \rho)$ and it is a b-map. Since ψ_1 is C^1-close to ψ, this composition is a diffeomorphism. □

1.2 The b-symplectic and the b-cotangent lifted Palais Theorems

We prove now the b-symplectic version of Palais Theorem, which is the b-symplectic analogue of the Symplectic Palais Theorem by Miranda in [MMZ12].

Theorem 1.5. *Let G be a compact Lie group and (M, Z, ω) a compact smooth b-symplectic manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow (M, Z, \omega)$ be two b-actions which are C^2-close. Then, there exists a b-symplectomorphism that conjugates ρ_1 and ρ_2, making them equivalent.*

Proof. Let G be a compact Lie group and (M, Z, ω) a compact smooth manifold. Let $\rho_1, \rho_2 : G \times M \longrightarrow M$ be two b-actions and assume that they are C^2-close. By Theorem 1.4, there exists a diffeomorphism φ that conjugates ρ_1 and ρ_2 and such that it is a b-map.

Set $\omega_0 = \omega$ and $\omega_1 = \varphi^*(\omega_0)$ and consider the linear path of b-symplectic structures

$$\omega_t = t\omega_1 + (1 - t)\omega_0, \quad t \in [0, 1],$$

which is a path of b-symplectic structures since ω_0 and ω_1 are close. We want to see that this path, which takes ω_0 to ω_1, is invariant respect to the action ρ_1.

By the Theorem 1.4 the b-diffeomorphism φ belongs to the arc-connected component of the identity, making it possible to construct an homotopy φ_t from $\varphi_0 := id$ to $\varphi_1 := \varphi$.

Then, we define a b-De Rham homotopy operator Q following the recipe given in [GS77] by Guillemin-Sternberg (see also [Can01]) which states the following. Suppose that ω_t is a smooth family of b-k-forms and that φ_t represents a one-parameter family of local diffeomorphisms such that $\varphi_t = id$ and $d\varphi_t/dt = X_t \circ \varphi_t$, i.e., φ_t is the flow of the b-vector field X_t. Then,

$$\frac{d}{dt}(\varphi_t^* \omega_t) = \varphi_t^* \left(\mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt} \right). \tag{1.4}$$

Fixing $\omega_t = \omega$ in Equation 1.4 and integrating over $t \in [0,1]$, we obtain:

$$\varphi_1^* \omega - \varphi_0^* \omega = \int_0^1 \varphi_t^* (\mathcal{L}_{X_t} \omega_t) dt. \tag{1.5}$$

Applying Cartan's formula, we get to the following equality:

$$\varphi_1^* \omega - \varphi_0^* \omega = \int_0^1 \varphi_t^* (\iota_{X_t} d\omega + d\iota_{X_t} \omega) dt = \tag{1.6}$$

$$= \int_0^1 \varphi_t^* (\iota_{X_t} d\omega) dt + d \int_0^1 \varphi_t^* (\iota_{X_t} \omega) dt. \tag{1.7}$$

Now, using Equation 1.7, we define the following b-De Rham operator Q:

$$Q(\omega) = \int_0^1 \varphi_t^* (\iota_{X_t} \omega) dt,$$

where the b-vector field X_t is defined by the isotopy φ_t. Equation 1.7 applied to $\omega = \omega_0$ tells that:

$$\omega_1 - \omega_0 = Q(d\omega) + dQ(\omega). \tag{1.8}$$

Since ω is a b-symplectic form, $Q(d\omega) = Q(0) = 0$ and:

$$\omega_1 - \omega_0 = dQ(\omega),$$

which proves that ω_0 and ω_1 belong to the same cohomology class and explicitly shows that $\omega_1 - \omega_0 = d\alpha$ for the b-1-form $\alpha = Q(\omega)$.

Now, let X_t be the b-vector field that satisfies

$$\iota_{X_t} \omega_t = -\alpha. \tag{1.9}$$

Notice that X_t is a b-vector field for any t, since α is a b-1-form and ω_t is a b-2-form for any t. Then, X_t will preserve (M, Z). Consider the averaged vector field of X_t with respect to a Haar measure $d\mu$ on G:

$$X_t^G := \int_G \rho_1(g)_* (X_t) d\mu. \tag{1.10}$$

Since the b-diffeomorphism φ conjugates the actions ρ_1 and ρ_2, which both preserve the initial b-symplectic form ω_0, the path of b-symplectic forms ω_t is invariant under ρ_1. Then, the b-vector field X_t^G satisfies the equation

$$i_{X_t^G}\omega_t = -\int_G \rho_1(g)^*(\alpha)d\mu,$$

which can be considered an averaging of Equation 1.9. Then, the invariant b-1-form defined by $\alpha_G = \int_G \rho_1(g)^*(\alpha)d\mu$ satisfies $\omega_1 - \omega_0 = d\alpha_G$ because the path ω_t is invariant under ρ_1.

Finally, consider the equation

$$X_t^G(\phi_t^G) = \frac{\partial \phi_t^G}{\partial t}.$$

At this point, there is a loss of one degree of differentiability with respect to the degree of differentiability of φ, but the existence of ϕ_t^G for all $t \in [0,1]$ is clear, because the manifold is compact and by 1.4 the conjugating b-diffeomorphism φ is of class $\mathcal{C}^2$.

Then, the flow ϕ_t^G commutes with the action of G given by ρ_1 and satisfies $\phi_t^{G*}(\omega_t) = \omega_0$ for all $t \in [0,1]$. In particular, at $t = 1$, we have that ϕ_1^G takes ω_1 to ω_0 in an equivariant way. □

We finally prove twisted b-cotangent lift version of the Palais Theorem. The twisted b-cotangent lift is appropriate here because when we want to eventually apply this result to integrable systems, these would be b-integrable Hamiltonian systems defined by b-functions. And the logarithm appearing on the local expression of the twisted b-1-form, i.e.:

$$\lambda_{tw} = \log|y_1|dx_1 + \sum_{i=2}^{n} y_i dx_i,$$

is compatible with the logarithm term of a b-function. On the contrary, the canonical b-cotangent lift is based on the singularity of the type $1/x$ appearing in the canonical b-1-form, and it does not produce a proper b-integrable Hamiltonian system.

Proposition 1.6. *Let G be a compact Lie group and (M,Z) a compact smooth b-manifold. Let $\rho_1, \rho_2 : G \times (M,Z) \longrightarrow (M,Z)$ be two b-actions which are C^1-close. Let $\hat\rho_1, \hat\rho_2 : G \times ({}^bT^*M, \omega) \longrightarrow ({}^bT^*M, \omega)$ be the twisted b-cotangent lifts of ρ_1, ρ_2, respectively. Then, there exists a b-symplectomorphism that conjugates $\hat\rho_1$ and $\hat\rho_2$, making them equivalent.*

Before proving this proposition, we prove that if two b-actions are C^1-equivalent, so are their twisted b-cotangent lifts and so are the induced moment maps.

Proposition 1.7. *Let G be a Lie group and let (M, Z) be a smooth manifold. Let $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ be two b-actions which are C^1-equivalent via a conjugation through a diffeomorphism φ which is a b-map. Let $\hat{\rho}_1, \hat{\rho}_2$ be the twisted b-cotangent lifts of ρ_1, ρ_2, respectively. Then, $\hat{\rho}_1$ and $\hat{\rho}_2$ are C^1-equivalent via the conjugation through the b-map $\hat{\varphi}$. The moment maps induced by $\hat{\rho}_1, \hat{\rho}_2$, denoted respectively by μ_1, μ_2, are equivalent via the conjugation through $\hat{\varphi}$.*

Proof. Assume $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ are two C^1-equivalent Lie group b-actions. Let φ be the C^1-b-diffeomorphism conjugating the two actions, i.e, let φ be a b-diffeomorphism such that $\rho_1 \circ \varphi = \varphi \circ \rho_2$.

Define $\hat{\varphi}(q, p) := (\varphi(q), ((d\varphi_q)^*)^{-1}(p))$, which is a b-diffeomorphism and can be thought as the twisted b-*cotangent lift of* φ. Consider the twisted b-cotangent lift of the actions ρ_1 and ρ_2, i.e. $\hat{\rho}_1$ and $\hat{\rho}_2$. By definition, $\hat{\rho}_i(q, p) = (\rho_i(q), ((d\rho_{i,q})^*)^{-1}(p))$. Then, we deduce that $\hat{\rho}_1 \circ \hat{\varphi} = \hat{\varphi} \circ \hat{\rho}_2$, and we conclude that the twisted b-cotangent lifts of the actions are equivalent on the cotangent bundle via conjugation by $\hat{\varphi}$, which is precisely the twisted b-cotangent lift of the diffeomorphism φ that conjugates ρ_1 and ρ_2 on the base.

Also, the moment maps induced by the twisted b-cotangent lifts of ρ_1 and ρ_2 are equivalent. □

Proof of Proposition 1.6. Let G be a compact Lie group and (M, Z) a compact smooth b-manifold. Let $\rho_1, \rho_2 : G \times (M, Z) \longrightarrow (M, Z)$ be two b-actions and assume that they are C^1-close. By Theorem 1.4, there exists a diffeomorphism φ that conjugates ρ_1 and ρ_2 and is a b-map.

Consider $\hat{\rho}_1, \hat{\rho}_2 : G \times ({}^bT^*M, \omega) \longrightarrow ({}^bT^*M, \omega)$, the twisted b-cotangent lifts of ρ_1 and ρ_2, respectively. By Proposition 1.7, the diffeomorphism $\hat{\varphi}$ conjugates $\hat{\rho}_1$ and $\hat{\rho}_2$ and it is also a b-map. To prove that the actions $\hat{\rho}_1$ and $\hat{\rho}_2$ are not only equivalent, but b-symplectically equivalent, we need to check that $\hat{\varphi}$ preserves the b-symplectic form. It preserves the canonical b-1-form λ of ${}^bT^*M$ and, hence, it preserves the b-symplectic form ω. □

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