Cotangent models for non-degenerate singularities

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Abstract

We work out cotangent lift models for integrable systems with non-degenerate singularities. We use different versions of the cotangent lift technique for different kind of singularities (in the sense of Williamson).

1 Introduction. Three elementary motivating examples

Hamiltonian integrable systems with non-degenerate singularities are widely found in Mechanics problems. Three of the most classical examples are the Harmonic Oscillator, the Simple Pendulum and the Spherical Pendulum.

1.1 The harmonic oscillator

Consider an ideal one-dimensional oscillating system consisting of a mass m connected to a rigid foundation by way of a spring of stiffness constant k, with no friction of any kind and, hence, with no loss of mechanical energy. The Hamiltonian of the system is the sum of the kinetical and the elastic potential energies. In terms of the natural coordinates of the system, which are the position x and the velocity v of the mass, it writes as:

$$\hat{H}(x,v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{1.1}$$

Applying the symplectic transformation

$$\begin{cases} x = q \cdot \frac{1}{\sqrt[4]{k/m}} \\ v = p \cdot \sqrt[4]{k/m} \end{cases} \tag{1.2}$$

The Hamiltonian becomes

$$H(p,q) = \frac{1}{2}\sqrt{mk}\left(p^2 + q^2\right) \tag{1.3}$$

Dropping the physical constants, this Hamiltonian is exactly the normal form of a one-dimensional system with an elliptic singularity at the origin, the unique equilibrium point of the system.

1.2 The simple pendulum

The simple pendulum is another of the simplest models in classical mechanics. The most natural approximation to its formulation is the Newtonian setting, where we consider the forces and acting in the system formed by a mass m attached to an end of a rigid massless rod of length l which has the other end fixed. It is assumed that the mass moves in the vertical plane formed by the vertical direction and the initial position and, since the rod has fixed length, the natural coordinate is the angle $\theta \in [0, 2\pi)]$ with respect to the lower vertical equilibrium position. Newton's law states that the acceleration of the mass in the direction of motion, which is always perpendicular to the direction of the rod, is proportional to the total force in this direction of motion. Since the only force in this direction is the component of the gravity force, Newton's law reduces to:

$$ma_\perp = F_\perp \tag{1.4}$$

Taking into account that the acceleration is related to the angular coordinate through $a_\perp(\theta) = l\frac{\partial^2\theta}{\partial t^2}$ and that the force is also function of the angle through $F_\perp(\theta) = -mg\sin\theta$, where g is the gravity acceleration, the equation rewrites as the following 2nd order ODE:

$$\frac{\partial^2\theta}{\partial t^2} = -\frac{g}{l}\sin\theta \tag{1.5}$$

If we define $\rho := \frac{\partial\theta}{\partial t}$, (1.5) is equivalent to the Hamiltonian first order system of ODEs:

$$\begin{cases} \frac{\partial\theta}{\partial t} = \rho \\ \frac{\partial\rho}{\partial t} = -\frac{g}{l}\sin\theta \end{cases} \tag{1.6}$$

whose Hamiltonian is

$$\hat{H}(\theta,\rho) = \frac{\rho^2}{2} - \frac{g}{l}\cos\theta \tag{1.7}$$

The first equilibrium point of (1.6) is found at $\theta = \rho = 0$, and it is an stable point. Dropping out the physical constants, the Hamiltonian there has the normal form $\bar{H} = \frac{1}{2}(\rho^2 + \theta^2)$, which corresponds to an elliptic singularity like in the harmonic oscillator. We are more interested in the second equilibrium point, found at $\theta = \pi, \rho = 0$.

The Hamiltonian there can be expanded as:

$$H(\theta, \rho) = \frac{1}{2}\left(\rho^2 - \frac{g}{l}\theta^2\right) \tag{1.8}$$

Dropping the physical constants, this Hamiltonian is corresponds to the normal form of a one-dimensional system with a hyperbolic singularity at the origin.

1.3 The spherical pendulum

A classical physical example of a singularity of focus-focus type comes from the spherical pendulum. Consider a point of mass m attached to an end of a rigid massless rod of length l. Assume that the other end of the rod is fixed at the origin and that the mass can move freely as long as it remains attached to the rod. The mass can move, then, on a sphere of radius l. The natural phase space is the cotangent bundle T^*S^2.

Spherical coordinates are the natural setting to study the dynamics of the spherical pendulum, while Cartesian coordinates are more appropriated to analyze the focus-focus singularity. The position of the point of mass will be given by $\vec{r} = (x, y, z)$, with $\|\vec{r}\| = l$. The conjugate variable to $\vec{r}$ is the linear momentum of the point, $\vec{p} = (p_x, p_y, p_z) = m\dot{\vec{r}}$, which has to satisfy $\vec{r} \cdot \dot{\vec{p}}) = 0$ in order to be contained in the tangent space of the sphere.

The Hamiltonian of the system is

$$H(\vec{r}, \vec{p}) = \frac{\|\vec{p}\|^2}{2m} + mgl\frac{\vec{r} \cdot \hat{z}}{\|\vec{r}\|} \tag{1.9}$$

where g accounts for the gravity acceleration and $\hat{z}$ is the unit vector in the z direction. There is another conserved quantity, the angular momentum in te z direction: $L := L_z = xp_y - yp_x$. H and L satisfy $\{H, L\} = 0$ and are independent almost everywhere. Hence, they form the Liouville integrable system corresponding the spherical pendulum.

There are two singularities in the pendulum system, corresponding to $z=-l$, so $\vec{r}_{-1}=(0,0,-l)$, and to $z=l$, so $\vec{r}_1=(0,0,l)$. We focus on $\vec{r}_1$, the unstable equilibrium. To study the system near $z=l$, we use local coordinates $(x,y,z)=(x,y,\sqrt{l^2-x^2-y^2})$. The conjugate momentum $\vec{p}=(p_x,p_y,p_z)$ satisfies locally that $p_z=0$. In these symplectic coordinates $\omega=dx\wedge dp_x+dy\wedge dp_y$ and the Hamiltonian becomes

$$H=\frac{1}{2ml^2}\left(p_x^2(l^2-x^2)+p_y^2(l^2-y^2)-2xyp_xp_y\right)+mg(\sqrt{l^2-x^2-y^2}-l) \tag{1.10}$$

At this point, it is convenient to apply a symplectic scaling in order to adimensionalize the Hamiltonian. We apply the transformation

$$\begin{cases} x=\frac{\xi}{\sqrt{m\nu}} \\ p_x=p_\xi\sqrt{m\nu} \\ y=\frac{\eta}{\sqrt{m\nu}} \\ p_y=p_\eta\sqrt{m\nu} \end{cases} \tag{1.11}$$

with $\nu=\sqrt{g/l}$. In these local symplectic coordinates near the unstable equilibrium of the spherical pendulum, $\omega=d\xi\wedge dp_\xi+d\eta\wedge dp_\eta$ and the Hamiltonian becomes:

$$H=\nu\left(\frac{1}{2}(p_\xi^2+p_\eta^2)-\frac{\kappa}{2}(\xi p_\xi+\eta p_\eta)^2+\frac{1}{\kappa}(\sqrt{1-\kappa\rho^2}-1)\right) \tag{1.12}$$

where $\rho^2=\xi^2+\eta^2$, $\nu^2=g/l$ and $1/\kappa=ml^2\nu=mgl/\nu$.

Now, the Williamson normal form at the unstable equilibrium of the spherical pendulum is achieved by the linear symplectic transformation

$$\sqrt{2}\xi=q_1-p_1,\quad \sqrt{2}p_\xi=q_1+p_1,\qquad \sqrt{2}\eta=q_2-p_2,\quad \sqrt{2}p_\eta=q_2+p_2\,.$$

The Hamiltonian in the new coordinates is:

$$H=\nu\left(p_1q_1+p_2q_2-\kappa\frac{1}{8}(q^2-p^2)^2+\frac{1}{\kappa}\sqrt{1-\kappa\rho^2}+\frac{\rho^2}{2}-\frac{1}{\kappa}\right),$$

where $p^2=p_1^2+p_2^2$, $q^2=q_1^2+q_2^2$ and $\rho^2=p^2/2+q^2/2-(p_1q_1+p_2q_2)$.

The quadratic part of the potential has been absorbed in the quadratic normal form terms $H_2=\nu(p_1q_2+p_2q_2)$. The remaining terms of the potential are of order 4 and higher. The angular momentum in the new variables is $L=q_1p_2-q_2p_1$. So, the system $F=(H,L)$ has a singularity of focus-focus type.

2 Non-degeneracy. Normal forms and Morse Theory

Theorem 2.1 (Eliasson-Miranda-Zung)**.** *Let $m \in M$ be a non-degenerate singularity in an integrable system on M given by $f = (f_1, \dots, f_n)$. Then, there exist symplectic local coordinates $(q,p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ on an open neighbourhood $U \subset M$ of m and a function $g = (g_1, \dots, g_n) : U \to \mathbb{R}^n$ such that its components are of one of the following forms:*

- *Elliptic:* $g_j(q,p) = q_j^2 + p_j^2$
- *Hyperbolic:* $g_j(q,p) = q_j p_j$
- *Focus-focus:* $g_j(q,p) = q_j p_{j+1} - q_{j+1} p_j \;\; g_{j+1}(q,p) = q_j p_j + q_{j+1} p_{j+1}$
- *Regular:* $g_j(q,p) = p_j$

and m corresponds to the origin $(q,p) = (0,0)$ and $\{f_i, g_j\} = 0$ for all i,j.

3 The cotangent lift

Let M be a differential manifold and T^*M its cotangent bundle. There is a canonically linear form λ on T^*M defined intrinsically by

$$\langle \lambda_p, v \rangle = \langle p, d\pi_p v \rangle \qquad for \qquad p \in T^*M, v \in T(T^*M)_p$$

where π is the canonical projection. In local coordinates $q_1, \dots, q_n, p_1, \dots, p_n$, the form is written as $\lambda = p_1 dq_1 + \cdots p_n q_n$ and is called the *Liouville 1-form.* Then, $\omega = -d\lambda$ is a symplectic form on T^*M, meaning that T^*M has a symplectic structure.

Definition 3.1. Let $\rho : G \times M \to M$ be a group action of a Lie group G on a smooth manifold M. For each $g \in G$, there is an induced diffeomorphism $\rho_g : M \to M$. The *cotangent lift* of ρ_g, denoted by $\hat{\rho}_g$, is diffeomorphism on T^*M given by

$$\hat{\rho}_g(q,p) := (\rho_g(q), ((d\rho_g)_q^*)^{-1}(p)), \qquad (q,p) \in T^*M$$

which makes the following diagram commute:

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

Example 3.2. Let $\rho : (\mathbb{R}^3, +) \times \mathbb{R}^3 \to \mathbb{R}^3$ be the Lie group action corresponding to a space translation defined by $\rho_x(q) = q + x$. Write (q, p) for an element of the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$.

By definition, $\hat{\rho}_x$, the cotangent lift of ρ_x is

$$\begin{aligned} \hat{\rho}_x(q, p) = (\rho_x(q), ((d\rho_x)_q^*)^{-1}(p)) = \\ = (q + x, ((Id^*)^{-1}(p)) = (q + x, p) \end{aligned}$$

Example 3.3. Let $\rho : SO(3, \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Lie group action defined by $\rho_A(q) = Aq$. Write (q, p) for an element of $T_q^*\mathbb{R}^3$. By definition, $\hat{\rho}_A$, the cotangent lift of ρ_A is

$$\hat{\rho}_A(q, p) = (\rho_A(q), ((d\rho_A)_q^*)^{-1}(p)) = (Aq, ((A^*)^{-1}(p)) = (Aq, Ap),$$

where the last equality holds because A is orthogonal. CHECK the cotangent lift of the angular momentum

4 A Mathematical perspective. Non-degenerate singularities as cotangent lifts

INTRO integrable systems

INTRO symplectic geometry

The cotangent lift for a regular value of the moment map (where A-L-M Theorem can be applied) is done in Miranda-Karshon POSAR THM i REF.

In the classical models of the harmonic oscillator, the simple pendulum and the spherical pendulum one already finds the three different types of non-degenerate singularities of Theorem 2.1 in its lowest dimensional case. A simple elliptic singularity is appears in the harmonic oscillator (REFEQ), a simple hyperbolic singularity shows up in the simple pendulum (REFEQ) and a simple focus-focus singularity arises in the spherical pendulum (REFEQ).

4.1 Cotangent lift of a double hyperbolic singularity

Take coordinates (x_1, x_2, y_1, y_2) such that the symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the moment map is $F = (f_1, f_2) = (x_1 y_1, x_2 y_2)$.

If we compute the Hamiltonian vector field associated to f_1 and f_2, we obtain

$$X_1 = -\frac{\partial f_1}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_1}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_1}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{4.1}$$

$$= -x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} = (-x_1, 0, y_1, 0) \tag{4.2}$$

and

$$X_2 = -\frac{\partial f_2}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_2}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{4.3}$$

$$= -x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2} = (0, -x_2, 0, y_2) \tag{4.4}$$

Now, consider the following action:

$$\begin{aligned} \rho\colon \quad (\mathbb{R}\times\mathbb{R})\times\mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ ((s,t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) &\longmapsto \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The differential of this action is computed in the following way. Consider $\gamma(r) = (x_1, x_2) + r(y_1, y_2)$. Then:

$$d\rho\colon T\mathbb{R}^2 \to T\mathbb{R}^2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \frac{d}{dr}(\rho\circ\gamma)\mid_{r=0} = \frac{d}{dr}\begin{pmatrix} e^{-s}(x_1 + ry_1) \\ e^{-t}(x_1 + ry_1) \end{pmatrix}\mid_{r=0} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then, $((d\rho)^*)^{-1}$ acts as:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{s} & 0 \\ 0 & e^{t} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

And the cotangent lift $\hat{\rho}$ associated to the group action is exactly

$$\hat{\rho}\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2 \tag{4.5}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{-s}x_1 \\ e^{-t}x_2 \\ e^{s}y_1 \\ e^{t}y_2 \end{pmatrix} \tag{4.6}$$

Deriving the vector with respect to s and evaluating at $(s = 0, t = 0)$, we obtain exactly X_1, while deriving the vector with respect to t and evaluating at $(s = 0, t = 0)$, we obtain exactly X_2, the vector fields associated to the hyperbolic singularity.

4.2 Cotangent lift of an elliptic singularity

The cotangent lift in the elliptic case uses a complex moment map which is not holomorphic. It is a formal development and by no means holomorphicity is assumed.

Take complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$ such that the symplectic form is $\omega = dz \wedge d\bar{z}$. The moment map in the elliptic case is $F = f_1 = x^2 + y^2 = z\bar{z}$.

The Hamilton's equations in this complex setting are:

$$\iota_X \omega = -df \iff \iota_{a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \bar{z}}} dz \wedge d\bar{z} = -\frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial \bar{z}} d\bar{z} \iff \begin{cases} a = -\frac{\partial f}{\partial \bar{z}} \\ b = \frac{\partial f}{\partial z} \end{cases} \tag{4.7}$$

If we compute the Hamiltonian vector field associated to f_1, we obtain

$$X_1 = -\frac{\partial f_1}{\partial \bar{z}} \left(\frac{\partial}{\partial z} \right) + \frac{\partial f_1}{\partial z} \left(\frac{\partial}{\partial \bar{z}} \right) \tag{4.8}$$

$$= -z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = (-z, \bar{z}) \tag{4.9}$$

Now, consider the following action, which is the same that we used for the hyperbolic cotangent lift but in complex coordinates:

$$\begin{aligned} \rho \colon \mathbb{R} \times \mathbb{C} &\to \mathbb{C} \\ (t, z) &\mapsto e^{-t} z \end{aligned}$$

The differential of this action is computed in the following way. Consider $\gamma(r) = z + r\bar{z}$. Then:

$$\begin{aligned} d\rho \colon T\mathbb{C} &\to T\mathbb{C} \\ \bar{z} &\mapsto \frac{d}{dr}(\rho \circ \gamma) \mid_{r=0} = \frac{d}{dr} e^{-t}(z + r\bar{z}) \mid_{r=0} = e^{-t} \cdot \bar{z} \end{aligned}$$

Then, $((d\rho)^*)^{-1}$ acts as

$$\bar{z} \longmapsto e^t \cdot \bar{z}$$

And the cotangent lift $\hat{\rho}$ associated to the group action is exactly

$$\hat{\rho} \colon T^*\mathbb{C} \to T^*\mathbb{C} \tag{4.10}$$

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} e^{-t} z \\ e^t \bar{z} \end{pmatrix} \tag{4.11}$$

Deriving the vector with respect to t and evaluating at $t = 0$, we obtain exactly X_1, the vector field associated to the elliptic singularity.

4.3 Cotangent lift of a focus-focus singularity

In a singularity of focus-focus type in a manifold of dimension 4, we can take coordinates (x_1, x_2, y_1, y_2) in a way that the symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the moment map is $F = (f_1, f_2) = (x_1y_2 - x_2y_1, x_1y_1 + x_2y_2)$.

If we compute the Hamiltonian vector field associated to f_1 and f_2, we obtain

$$X_1 = -\frac{\partial f_1}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_1}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_1}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{4.12}$$

$$= x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} = (x_2, -x_1, y_2, -y_1) \tag{4.13}$$

and

$$X_2 = -\frac{\partial f_2}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_2}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{4.14}$$

$$= -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = (-x_1, -x_2, y_1, y_2) \tag{4.15}$$

Let $G = S^1 \times \mathbb{R}$ and $M = \mathbb{R}^2$. Consider the action of a rotation and a radial dilation of $\mathbb{R}^2$ given by

$$\rho\colon (S^1 \times \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$((\theta, t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) \mapsto \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The differential of this group action at a point (x_1, x_2) is the following linear map

$$d\rho\colon T\mathbb{R}^2 \to T\mathbb{R}^2 \tag{4.16}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto e^{-t} \begin{pmatrix} y_1\cos\theta + y_2\sin\theta \\ -y_1\sin\theta + y_2\cos\theta \end{pmatrix} \tag{4.17}$$

Then, $((d\rho)^*)^{-1}$ acts as:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto e^{t} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

And the cotangent lift $\hat{\rho}$ associated to the group action is exactly

$$\hat{\rho}\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2 \tag{4.18}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{-t}(x_1 \cos\theta + x_2 \sin\theta) \\ e^{-t}(-x_1 \sin\theta + x_2 \cos\theta) \\ e^{t}(y_1 \cos\theta + y_2 \sin\theta) \\ e^{t}(-y_1 \sin\theta + y_2 \cos\theta) \end{pmatrix} \tag{4.19}$$

Deriving the vector with respect to θ and evaluating at 0 we obtain exactly $X_1 = (x_2, -x_1, y_2, -y_1)$. While deriving the vector with respect to t and evaluating at 0 we obtain exactly $X_2 = (-x_1, -x_2, y_1, y_2)$.

5 Reformulating Eliasson-Vey Theorem

Theorem 5.1. *Any integrable system with a non-degenerate singularity is equivalent in a neighbourhood of the singularity to the integrable system defined by a cotangent lift with the cotangent symplectic structure. This cotangent lift can be a combination of blocks of the following type, depending on the kind of singularity of the different components of the moment map:*

1. *Standard cotangent lift for hyperbolic singularities, corresponding to the components of the form*
$$f_i = x_i y_i,$$
2. *Complexified cotangent lift for elliptic singularities, corresponding to the components of the form*
$$f_i = x_i^2 + y_i^2,$$
3. *Cotangent lift of the complexification of a compact group for focus-focus singularities, corresponding to the pairs of components of the form*
$$f_i = x_i y_{i+1} - x_{i+1} y_i, \quad f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$$

6 Complexification of a Lie group

We have the following definition of the complexification of a compact Lie group.

Definition 6.1. Let K be a compact Lie group. An *analytic complexification* of K is a complex analytic group G together with a Lie group homomorphism $i : K \longrightarrow G$ such that, if $f : K \longrightarrow H$ is another Lie group homomorphism into a complex analytic group H, then there exists a unique analytic homomorphism $F : G \longrightarrow H$ such that $f = F \circ i$.

In the same way, we can consider the complexification of a Lie algebra, which is easier to define because it is only the complexification of a real vector space. To get from a real Lie algebra representation to a complex one, we extend the action of real scalars to complex scalars. In the case of real matrices, complexification is essentially allowing complex coefficients and using the same rules for multiplying matrices as before.

Definition 6.2. The complexification $V^{\mathbb{C}}$ of a real vector space V is the space of pairs (v_1, v_2) of elements of V with product by $a + ib \in \mathbb{C}$ given by

$$(a + ib)(v_1, v_2) = (av_1 - bv_2, av_2 + bv_1)$$

This definition makes it possible to think of the complexification of V as $V^{\mathbb{C}} = V + iV$. Now, for any real Lie algebra $\mathfrak{g}$, the complexification $\mathfrak{g}^{\mathbb{C}}$ is the set of pairs of elements (X, Y) of $\mathfrak{g}$, with the usual rule for the product by complex scalars, which can be thought of as $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$.

The Lie bracket on $\mathfrak{g}$ extends in a natural way to a Lie bracket on $\mathfrak{g}^{\mathbb{C}}$ by:

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2] - [Y_1, Y_2], [X_1, Y_2] + [Y_1, X_2]),$$

which can be thought as the following computation:

$$[X_1 + iY_1, X_2 + iY_2] = [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2])$$

Example 6.3. The Lie group $G = \mathbf{GL}(n, \mathbb{R})$ has the Lie algebra $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ of real $n \times n$ matrices. Its complexification is nothing else than $\mathfrak{gl}(n, \mathbb{C})$, since $\mathfrak{gl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{R}) + i\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{C})$.

Example 6.4. The Lie group $U(n)$ has the Lie algebra $\mathfrak{u}(n) \subset \mathfrak{gl}(n, \mathbb{C})$ of anti-Hermitian matrices. Since the product of the anti-Hermitian matrices by i gives the Hermitian matrices, the complexification $\mathfrak{u}(n)^{\mathbb{C}}$ of $\mathfrak{u}(n)$ is exactly $\mathfrak{gl}(n, \mathbb{C})$.

Remark 6.5. With these two examples, one can see that different Lie algebras can have the same complexification.

Example 6.6. The Lie groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ have the same Lie algebra, since $SO(n, \mathbb{R})$ is the connected component of $O(n, \mathbb{R})$ that contains the identity. The complexification of the Lie algebra $\mathfrak{so}(n, \mathbb{R})$ of the real anti-symmetric matrices is naturally the Lie algebra of the complex anti-symmetric matrices $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$, since $\mathfrak{so}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{so}(n, \mathbb{R}) + i\mathfrak{so}(n, \mathbb{R}) = \mathfrak{so}(n, \mathbb{C})$.

The topology of the simple orthogonal group over the complex numbers is quite simple. As well as $SO(n, \mathbb{R})$, $SO(n, \mathbb{C})$ is a connected Lie group, since any element can be joined by a path to the identity. The elements of $SO(n, \mathbb{C})$

can be thought as rotations and can be identified in a hyperbolic basis with the invertible elements of $\mathbb{C}$, i.e., with $\mathbb{C} \setminus \{0\}$. The topology of this set can be, at its turn, identified to the Cartesian product $S \times \mathbb{R}$.

6.1 The focus-focus singularity as a complexified model

The complexification of $S^1 \cong SO(n, \mathbb{R})$ gives $SO(n, \mathbb{C}) \cong S^1 \times \mathbb{R}$. The cotangent model of the focus-focus singularity in Example 4.3, based on the cotangent lift of the action of the non-compact Lie group $S^1 \times \mathbb{R}$, can be now seen as the cotangent lift of the complexification of the compact Lie group S^1.

Give an interpretation to the complex vector field given by $\dot{X} = AX$, with $A \in SO(2, \mathbb{C})$.

6.2 The quaternionic approach

The quaternionification of the cotangent lift can be created through the change:

$$q_i = x_i + x_{i+1}i + y_i j + y_{i+1}k = \underbrace{x_i + x_{i+1}i}_{\alpha} + \underbrace{(y_i + y_{i+1}i)}_{\beta} j \tag{6.1}$$

where we used that $ij = k$.

Then, the conjugate is

$$\bar{q}_i = x_i - x_{i+1}i - y_i j - y_{i+1}k = \bar{\alpha} - \bar{\beta} j$$

Now, compute

$$dq_i \wedge d\bar{q}_i = (dx_i + dx_{i+1}i + dy_i j + dy_{i+1}k) \wedge (dx_i - dx_{i+1}i - dy_i j - dy_{i+1}k) = \tag{6.2}$$

$$= 2(dx_{i+1} \wedge dx_i + dy_{i+1} \wedge dy_i)i+ \tag{6.3}$$

$$+2(dy_i \wedge dx_i + dx_{i+1} \wedge dy_{i+1})j+ \tag{6.4}$$

$$+2(dy_{i+1} \wedge dx_i + dy_i \wedge dx_{i+1})k \tag{6.5}$$

Take ω as

$$\begin{aligned}\omega =& \frac{1}{2} dq_i \wedge d\bar{q}_i = \\ =& (dx_{i+1} \wedge dx_i + dy_{i+1} \wedge dy_i)i+ \\ &+(dy_i \wedge dx_i + dx_{i+1} \wedge dy_{i+1})j+ \\ &+(dy_{i+1} \wedge dx_i + dy_i \wedge dx_{i+1})k\end{aligned}$$

which lives in the space of the unit quaternions (does not have real part).

6.3 The octonions approach

Following [Fur12], we introduce octonions in order to take advantage of its algebraic properties to build a model for the cotangent bundle of a high-dimensional complexified manifold.

The $e_1, \ldots, e_7$ are the octonionic imaginary units $\left(e_n^2 = -1\right)$, apart from the real $e_0 = 1$, which multiply according to Figure 1. Any three imaginary units on a directed line segment in Figure 1 act as if they were a quaternionic triple. For instance, $e_6 e_1 = -e_1 e_6 = e_5$, $e_1 e_5 = -e_5 e_1 = e_6$, $e_5 e_6 = -e_6 e_5 = e_1$, $e_4 e_1 = -e_1 e_4 = e_2$, etc. Octonionic multiplication harbours various symmetries, such as index doubling symmetry: $e_i e_j = e_k \Rightarrow e_{2i} e_{2j} = e_{2k}$, which can be seen by rotating Figure 1 by 120 degrees.

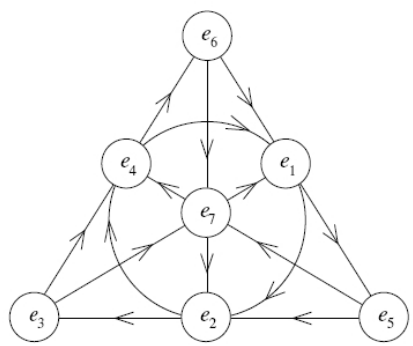

Figure 1: Octonionic multiplication rules