Master course on *Differentiable Manifolds*

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Chapter 1

Introduction to Differential Topology

In this chapter we mainly follow [GP74].

1.1 Basic tools in Differential Topology

1.1.1 A crash course on Differential Geometry

Definition 1.1.1 (Abstract smooth manifold)**.** A *smooth manifold* is a two-countable Hausdorff topological space X such that, for every $p \in X$, there exists an open neighbourhood $U \subset X$ and a mapping $\varphi : U \to \mathbb{R}^n$ which induces an homeomorphism between U and $\varphi(U)$ and such that given intersection U_i and U_j, the mapping

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is $\mathcal{C}^\infty$. The dimension of the manifold is n.

Remark 1.1.2. We will usually assume $X \subseteq \mathbb{R}^n$

Theorem 1.1.3 (Whitney)**.** *Every manifold X of dimension k can be embedded into $\mathbb{R}^{2k}$.*

Definition 1.1.4 (Smooth manifold)**.** A *smooth manifold* is a two-countable Hausdorff topological space X such that, for every $p \in X$, there exists $\varphi : U \to \mathbb{R}^k$, where U is an open neighbourhood of p (with the induced euclidean topology) and φ is a local diffeomorphism.

Remark 1.1.5. A diffeomorphism is a smooth map with smooth inverse.

Remark 1.1.6. We will use the following notation:
$\varphi : U_i \subset X \subset \mathbb{R}^n \to V_j \subset \mathbb{R}^k$ are coordinate charts.
$\phi := \varphi^{-1} : V_j \to U_i$ are called parametrizations.

Example 1.1.7. The circle, $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$, is a smooth manifold of dimension 1. It can be equipped with the following charts. Any point $z \in S^1$ can be written as $z = e^{2i\pi c}$ for a unique $c \in [0, 1)$. Define the map

$$\begin{array}{rccc} \nu_z : & \mathbb{R} & \longrightarrow & S^1 \\ & t & \longmapsto & e^{2i\pi t} \end{array}$$

For any c, the map ν_z restricted to the interval $I_c = (c-1/2, c+1/2)$, namely $\mu_z = \nu_z|_{I_c}$ is a homeomorphism from I_c to $S^1 \setminus \{-z\}$, which is, in particular, a neighbourhood of z. Then, $\varphi_z := \mu_z^{-1}$ is a chart of S^1 near z.

Example 1.1.8. A generalization of the previous example, the sphere $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\} \subset \mathbb{R}^{n+1}$. Two proper charts for S^n are the North and South *stereographic projections*, φ_N and φ_S:

$$\begin{array}{rccc} \varphi_N : & S^n \setminus \{(-1, 0, \ldots, 0)\} & \longrightarrow & \mathbb{R}^n \\ & (x_0, x_1, \ldots, x_n) & \longmapsto & (1 + x_0)^{-1} \cdot (x_1, \ldots, x_n) \end{array}$$

$$\begin{array}{rccc} \varphi_S : & S^n \setminus \{(+1, 0, \ldots, 0)\} & \longrightarrow & \mathbb{R}^n \\ & (x_0, x_1, \ldots, x_n) & \longmapsto & (1 - x_0)^{-1} \cdot (x_1, \ldots, x_n) \end{array}$$

Example 1.1.9. The Cartesian product $X \times Y$ of two manifolds X and Y is a manifold. If $\{U_\alpha, \varphi_\alpha\}$ and $\{V_\beta, \psi_\beta\}$ are atlases for the manifolds X and Y of dimensions m and n, respectively, then the collection $\{U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}$ of charts is an atlas on $X \times Y$.

Example 1.1.10. The n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$ can be equipped with the Cartesian product of charts of S^1, i.e., with the charts $\{(\varphi_{z_1}, \ldots, \varphi_{z_n})\}$ where each φ_i is a chart of S^1.

Example 1.1.11. An open subset $U \subset X$ of a manifold is also a manifold. Its charts can be taken as restrictions $\varphi|_U$ of charts φ for M.
For instance, the real $n \times n$ matrices, $\mathrm{Mat}(n, \mathbb{R})$, form a manifold, which is a vector space isomorphic to $\mathbb{R}^{n^2}$. The subset $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \mid detA \neq 0\}$ is open. Hence it is a manifold.

Example 1.1.12. The real projective plane $\mathbb{RP}^n \cong S^n/_{x\sim -x} \cong S^n/(\mathbb{Z}/(2))$

Definition 1.1.13 (Canonical definition of the tangent space at a point)**.** The *tangent space* of the manifold X at the point $x \in X$ is $T_x(X) := \mathrm{Im}\, d\phi_0$, where $\phi_0(0) = \varphi_0^{-1}(0) = x$ (φ_0 is a *centered chart*).

Remark 1.1.14. This definition is canonical, does not depend on the parametrization. Take two different parametrizations ϕ^1 and ϕ^2 and define $h := (\phi^2)^{-1} \circ \phi^1$, which is smooth by definition. Then,

$$(d\phi^1)_0 = (d\phi^2)_0 \circ h_0,$$

so $\mathrm{Im}(d\phi^1)_0 \subseteq \mathrm{Im}(d\phi^2)_0$. Interchanging roles, we obtain that $\mathrm{Im}(d\phi^2)_0 \subseteq \mathrm{Im}(d\phi^1)_0$ and, hence, $\mathrm{Im}(d\phi^1)_0 = \mathrm{Im}(d\phi^2)_0$.

Remark 1.1.15. A consequence of Definition 1.1.13 is that the dimension of $T_x(X)$ is equal to the dimension of X.

Definition 1.1.16 (Derivative of an smooth mapping)**.** Let $f : X \to Y$ be a smooth map between smooth manifolds. Suppose $x \in X$ and $y = f(x)$. Then, $(df)_x : T_xX \to T_yY$ is defined as follows. Take any charts φ_i and ψ_j, centered at x and y, respectively.

$$\begin{tikzcd}
T_xX \arrow[r, "(df)_x"] \arrow[d, "\varphi_i"'] & T_yY \arrow[d, "\psi_j"] \\
\mathbb{R}^k \cong T_0\mathbb{R}^k \arrow[r, "h"] & \mathbb{R}^l \cong T_0\mathbb{R}^l
\end{tikzcd}$$

So $(df)_x := d\psi_j^{-1} \circ h \circ d\varphi_i$.

Remark 1.1.17. The chain rule holds.

1.1.2 Basics on Differential Geometry

Definition 1.1.18. A *curve* $\gamma(t)$ on a smooth manifold M is a differentiable map from $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ to M.

Definition 1.1.19. Let $f : M \to N$ be a differentiable map between smooth manifolds. The *linear tangent mapping* of $f : M \to N$ at $q \in M$, denoted by $(df)_q$, is defined as follows. If $\gamma'(0)$ is the tangent vector to the curve $\gamma(t) \in M$, $df_q : T_q(M) \to T_{f(q)}N$ assigns to it the tangent vector to the curve $f(\gamma(t)) \in N$, at $t = 0$.

This definition allows to draw the following commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Definition 1.1.20 (Vector field 1)**.** A *vector field* X over a manifold M is a derivation. That is, it is a $\mathbb{R}$-linear map $X : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})$ such that it satisfies the *Leibniz rule*, i.e. $X(f, g) = fX(g) + X(f)g$. A *vector* X_p at a point $p \in M$ satisfies $X_p(f, g) = f(p)X_p(g) + X_p(f)g(p)$.
The set of all vector fields over a manifold is $\mathfrak{X}(M)$.

Definition 1.1.21 (Vector field 2)**.** A *vector* X_p at a point $p \in M$ is an equivalence class $[\gamma]$ of paths $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = X_p$ and $\widetilde{\gamma} \sim \gamma$ if $d(\varphi \circ \gamma)/dt|_{t=0} = d(\varphi \circ \widetilde{\gamma})/dt|_{t=0}$ for any coordinate chart $(u_\alpha, \varphi_\alpha)$. The association of a vector X_p to each point $p \in M$ defined a *vector field*.

Definition 1.1.22 (Tangent bundle 1)**.** The *tangent bundle* TM of a manifold M is defined as $TM = \bigsqcup_{p \in M} T_pM = \{(p, X_p) \mid p \in M, X_p \in T_pM\}$, and is equipped with the natural projection $\pi : (p, X_p) \mapsto p : TM \to M$.

Remark 1.1.23. TM is a smooth manifold of dimension $2 \cdot \dim M$.

Definition 1.1.24 (Vector field 3)**.** A *vector field* X over a manifold M is a map $X : p \mapsto X_p : M \to TM$ such that $\pi \circ X = id_M$. In other words, a vector field is a *section* of the tangent bundle.

We introduce a definition that will be used later.

Definition 1.1.25 (Lie Algebra)**.** A *Lie algebra* is a pair $(\mathfrak{X}(M), [\cdot, \cdot])$ such that $\mathfrak{X}(M)$ is a vector space and such that the *Lie bracket* $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an operator which is skew-symmetric, bilinear and satisfies Jacobi identity ($[X, Y](f) = X(Y(f)) - Y(X(f))$).

Definition 1.1.26. An *integral curve* γ of a vector field $X \in \mathfrak{X}(M)$ is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(t) = X(\gamma(t))$.

Locally, integral curves always exist. Take coordinates $(U_\alpha, \varphi_\alpha = (x_1, \ldots, x_n))$ and $X = \sum_{i=1}^n X^i \partial/\partial x_i$. Then, the equality writes as

$$\sum_{i=1}^n \gamma'^i(t) \frac{\partial}{\partial x_i} = \sum_{i=1}^n X^i(\gamma(t)) \frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \gamma'^i(t) = X^i(\gamma(t)),$$

which is a system of ODE's, whose solution exists locally but not always globally.

Definition 1.1.27. A vector field $X \in \mathfrak{X}(M)$ is *complete* when all its integral curves exist globally (they can be extended to all $t \in \mathbb{R}$).

Exercise 1.1.28. If M is a compact manifold and $X \in \mathfrak{X}(M)$, then X is complete.

Definition 1.1.29 (Flow of a vector field)**.** Assume M is a compact manifold. Then, the *flow* ϕ of $X \in \mathfrak{X}(M)$ is given by

$$\begin{array}{rccc} \phi: & M \times \mathbb{R} & \longrightarrow & M \\ & (x,t) & \longmapsto & \gamma_x^X(t) \end{array}$$

where $\gamma_x^X(t)$ is an integral curve of X passing through x.

It is immediate to check, from the definition, that any flow ϕ satisfies the following properties:

- $\phi(x,0) = x$.
- $\forall t \in \mathbb{R}$, $\widetilde{\phi}_t(x) := \phi(x,t)$ is a diffeomorphism.
- $\phi(\phi(x,s),t) = \phi(x,s+t)$.

Consider two vector fields $X, Y \in \mathfrak{X}(M)$. With the idea of deriving Y with respect to X, one could naively write

$$\lim_{t \to 0} \frac{Y(\gamma_p^X(t)) - Y(p)}{t},$$

where $\gamma_p^X(t)$ is the flow of X, but this expression is not well defined since $Y(\gamma_p^X(t)) \in T_{\gamma_p^X(t)}M$ and $Y(p) \in T_pM$, so we can not subtract them.

Instead, we use the derivative of $\gamma_p^X(t)$, that fixes it since $d\gamma_p^X(t) : T_{\gamma_p^X(t)}M \to T_pM$. Then, we can write

$$\lim_{t \to 0} \frac{(d\gamma_p^X(-t))(Y(\gamma_p^X(t))) - Y(p)}{t},$$

which is well defined. This is how we define the *Lie derivative*.

Definition 1.1.30 (Lie derivative)**.** For any two vector fields $X, Y \in \mathfrak{X}(M)$, the *Lie derivative* of Y with respect to X is

$$\mathcal{L}_X Y := \frac{d}{dt}(d\gamma_p^X(-t))(Y(\gamma_p^X(t)))|_{t=0}.$$

The Lie bracket that appears in the definition of a Lie algebra (1.1.25) coincides with the definition of the Lie derivative, i.e. $[X,Y] = \mathcal{L}_X Y$.

If X is a smooth vector field on M, the application of df_q to X, denoted by $f_* X$, is called the *pushforward*.

The linear tangent mapping is also called the *differential* of f at $q \in M$. For a smooth vector field $X \in \mathfrak{X}(M)$, the differential acts on X exactly as $(df_q)(X) = X(f)$. It is more intuitive, though, to think that it is the directional derivative of f with respect to the field X. Notice that df_q is an element of $(T_qM)^*$, the dual of T_qM.

Definition 1.1.31. Given two vector fields $X, Y \in \mathfrak{X}(M)$, the *Lie bracket of vector fields* between X and Y is defined as the field $[X,Y]$ that assigns to each $q \in M$ the tangent vector given by

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

Definition 1.1.32. A *differential r-form* (or, simply, a r-form) α on a smooth manifold M consists on assigning to each $q \in M$ an element α_q of $\wedge^r(T_qM)^*$, where $\wedge^r$ means the *wedge product* of r dual vector spaces. The space of all r-forms on M is denoted by $\Omega^r(M)$.

k-forms satisfy $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$, where $|\alpha|$ is the degree of the form α.

When a differential form $\alpha \in \Omega^r(M)$ satisfies $d\alpha = 0$, it is called a *closed form.* If $\alpha = d\beta$, for some $\beta \in \Omega^{r-1}(M)$, it is called an *exact form.*

Definition 1.1.33. Let $X, X_2, \ldots, X_r \in \mathfrak{X}$ be smooth vector fields and $\alpha \in \Omega^r(M)$ an r-form. The *interior product* $\iota_X \alpha$ of α with X is a $r-1$-form that is defined as

$$\iota_X \alpha(X_2, \ldots, X_r) = \alpha(X, X_2, \ldots, X_r).$$

It is also called the *contraction* between α and X.

Definition 1.1.34. Let α be a differential r-form on a smooth manifold M. The *exterior derivative* of α is the differential $(r+1)$-form $d\alpha$ defined in the following way. If $X_0, X_1, \ldots, X_r$ are smooth vector fields defined on M, then

$$\begin{aligned} d\alpha(X_0, \ldots, X_r) = &\sum_i (-1)^i X_i(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_r)) + \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j \ldots, X_r), \end{aligned}$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields and $\hat{X}_i$ denotes the omission of the element X_i.

Axiomatically, the exterior derivative can be defined in the following way:

Theorem 1.1.35. *For each k, $\exists d : \Omega^k(M) \to \Omega^{k+1}(M)$ such that:*

- *d is $\mathbb{R}$-linear.*
- *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$.*
- *d coincides with the usual differential operator in $\Omega^0(M)$.*
- *If $f \in \Omega^0(M)$, then $d(df)) = 0$.*

The exterior derivative satisfies the following properties:

1. If $U \subset M$ is open, then $\alpha|_U = \beta|_U \Rightarrow d\alpha|_U = d\beta|_U$.
2. $d^2 = 0$.
3. $d(f dx_1 \wedge \cdots \wedge dx_n) = df \wedge dx_1 \wedge \cdots \wedge dx_n$.

Theorem 1.1.36. *Let M^n be a manifold. Let ∂M be the boundary of M and $i : \partial M \hookrightarrow M$ the inclusion. Let $\alpha \in \Omega^{n-1}(M)$ be a differential form with compact support. Then,*

$$\int_M d\alpha = \int_{\partial M} i^*\alpha.$$

The generalization of the Lie bracket between vector fields is the *Lie derivative*. In the more general definition, the Lie derivative $\mathcal{L}_X R$ evaluates the change of a tensor field R along the flow of a particular vector field X on a smooth manifold M. We list the three most used Lie derivatives:

- The Lie derivative of a scalar function $f \in \mathcal{C}(M)$ with respect to X is $\mathcal{L}_X f = X(f)$, the directional derivative of f with respect to the X field.
- The Lie derivative of a vector field $Y \in \mathfrak{X}(M)$ with respect to X is $\mathcal{L}_X Y = [X, Y]$, the Lie bracket.
- The Lie derivative of a r-form $\alpha \in \Omega^r(M)$ with respect to X is $\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha$, which is known as Cartan's magic formula.

Definition 1.1.37. Let M be a differential manifold. For a point $q \in M$, consider the set of all tangent vectors on M at q and denote it by T_qM. Then, TM, the *tangent bundle* of M, is defined as the disjoint union of all the sets of tangent vectors, i.e.:

$$TM := \bigsqcup_{q \in M} T_qM.$$

It is equipped with the canonical projection

$$\begin{array}{rccc} \pi: & TM & \longrightarrow & M \\ & (q,v) & \longmapsto & q \end{array}$$

In the same way, the *cotangent bundle* of M, T^*M, is defined as the dual vector bundle over M, dual to the tangent bundle TM of M. It is also equipped with the canonical projection

$$\begin{array}{rccc} \pi: & T^*M & \longrightarrow & M \\ & (q,p) & \longmapsto & q \end{array}$$

Definition 1.1.38. Let $\varphi : M \to N$ be a smooth map and let $f : N \to \mathbb{R}$ be a smooth function. The *pullback* of f by φ is a smooth map defined by

$$(\varphi^* f)(q) := f(\varphi(q)), \qquad q \in M.$$

Let $\alpha \in \Omega^r(N)$ be a differential r-form on N. Let $X_1, \ldots, X_r \in \mathfrak{X}(M)$ be smooth vector fields on M. The pullback of α by φ is a differential r-form defined by

$$(\varphi^*\alpha)_q(X_1, \ldots, X_r) := \alpha_{\varphi(q)}(d\varphi_q(X_1), \ldots, d\varphi_q(X_r)), \qquad q \in M.$$

Example 1.1.39. Consider the 1-form $\alpha = xdy - ydx$ and the usual change to polar coordinates:

$$\begin{array}{rccc} \varphi: & \mathbb{R} \times S^1 & \longrightarrow & \mathbb{R}^2 \\ & (r,\theta) & \longmapsto & (r\cos\theta, r\sin\theta) \end{array}$$

To compute $\varphi^*\alpha$, the pullback of α by φ, we have two methods:

Method 1.

$$\begin{aligned} (\varphi^*\alpha)_{(r,\theta)} &= r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta) = \\ &= r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta(\cos\theta\, dr - r\sin\theta\, d\theta) = \\ &= r^2\cos^2\theta\, d\theta + r^2\sin^2\theta\, d\theta = r^2\, d\theta \end{aligned}$$

Method 2.

$$(\varphi^*\alpha)_{(r,\theta)}(v_r, v_\theta) = \begin{pmatrix}-r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}v_r \\ v_\theta\end{pmatrix} =$$

$$= \begin{pmatrix}0 & r^2\end{pmatrix}\begin{pmatrix}v_r \\ v_\theta\end{pmatrix}$$

$\implies \varphi^*\alpha = r^2\,d\theta$

The pullback acts nicely on wedge product of differential forms, i.e., $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

The pullback also acts nicely with the exterior derivative, i.e., $f^*(d\alpha) = d(f^*\alpha)$.

Differential forms are a tool to measure k-volumes.

Example 1.1.40. Let M be a smooth manifold. Let $\alpha \in \Omega^1(M)$ be a 1-form and let $\gamma : I \to M$ be a differentiable curve on the manifold. Then, $\mathrm{Len}_\alpha(\gamma) = \int_I \gamma^*\alpha =: \int_\gamma \alpha$.

Example 1.1.41. Let M^n be a smooth manifold. Let $\omega \in \Omega^n(M)$ be a volume form and let (U, φ) be a local coordinate chart. Then, $\mathrm{Vol}_\omega(U) = \int_U \varphi^*\omega$. This, together with a partition of unity (see def 1.6.1), makes it possible to define $\int_M \omega$.

Example 1.1.42. If $\psi : S^k \to M^n$ is an immersion, then we can define $\int_S \alpha$ for an $\alpha \in \Omega^k(M)$.

1.2 De Rham Cohomology

1.3 Transversality and Normal Forms

1.3.1 The Inverse Function Theorem, Immersions, Submersions, the Regular Value Theorem

In many cases, new manifolds are created from considering *submanifolds*. Differential Topology offers a technique in this direction that consists in considering objects that cut *transversally*.

Theorem 1.3.1. *Suppose $f : X \to Y$ is a smooth map whose derivative at any point $x \in X$ is an isomorphism. Then, f is a local diffeomorphism.*

An immediate consequence of the theorem is that, under its hypotheses, $\dim X = \dim Y$. Going in the other direction, if we know that $\dim X < \dim Y$, the best we can get is that $f : x \to Y$ is an immersion. This is the case if, for every $x \in X$, $(df)_x$ is injective. If, on the contrary, $\dim X > \dim Y$, the best we can reach is a submersion, and this is the case if, for every $x \in X$, $(df)_x$ is exhaustive.

Theorem 1.3.2 (Local Immersion Theorem)**.** *Suppose $f : X \to Y$ is an immersion at a point $x \in X$, and let $y = f(x) \in Y$. Assume $\dim X = k < l = \dim Y$. Then, there exist local coordinates around x and y such that*

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd} \tag{1.3.1}$$

with $\phi(0) = x$ and $\psi(0) = y$.

We seek to modify g in order to apply the Inverse Function Theorem. As $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is injective, there exists a linear change of coordinates that transforms it into the matrix $\begin{pmatrix} I_k \\ 0 \end{pmatrix}_{l \times k}$. And define $G : U \times \mathbb{R}^{l-k} \to \mathbb{R}^k$ by

$$G(a, a_{l+1}, ..., a_k) = (g(a), a_{l+1}, ..., a_k) \longrightarrow dG_0 = \begin{pmatrix} dg_0 & 0 \\ 0 & dId_{l-k} \end{pmatrix} = Id_k. \tag{1.3.2}$$

So G is a local diffeomorphism at 0. By construction, $g = \pi \circ G$ (where π is the local submersion or projection in Euclidean spaces). Finally,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\pi"] & V \arrow[u, "\psi \circ G"'] \end{tikzcd} \tag{1.3.3}$$

□

Exercise 1.3.3. Is it true that the image of an immersion always a submanifold? Prove it or give a counterexample.

Definition 1.3.4. An immersion that is injective and proper is called an *embedding.*

Theorem 1.3.5 (Local Submersion Theorem)**.** *Suppose $f : X \to Y$ is a submersion at a point $x \in X$, and let $y = f(x) \in Y$. Then, there exist local coordinates around x and y such that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_l)$, where* $\dim X = k > l \dim Y$.

Proof. The proof of this theorem is just like proof of the local classification of immersions. Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\big\uparrow & & \psi\big\uparrow \\ U & \xrightarrow{g} & V \end{array} \tag{1.3.4}$$

with $\phi(0) = x$ and $\psi(0) = y$.

Since $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is surjective, there exists a linear change of coordinates that transforms it into the matrix $(I_l|0)_{l\times k}$. Define $G : U \to \mathbb{R}^k$ by

$$G(a, a_{l+1}, ..., a_k) = (g(a), a_{l+1}, ..., a_k) \longrightarrow dG_0 = \begin{pmatrix} dg_0 & 0 \\ 0 & dId_{l-k} \end{pmatrix} = Id_k. \tag{1.3.5}$$

So G is a local diffeomorphism at 0. Then G is locally invertible, and G^{-1} exists as diffeomorphism of some open neighborhood U', of 0, into U. By construction, $g = \pi \circ G$ (where π is the local submersion or projection in Euclidean spaces). Finally,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \phi\,\circ\,G^{-1}}\big\uparrow & & \big\uparrow{\scriptstyle\psi} \\ U' & \xrightarrow{\pi} & V \end{array} \tag{1.3.6}$$

We have shown that the canonical submersion π is equivalent to any submersion f.

□

Definition 1.3.6. The *codimension* of an arbitrary submanifold Z of a manifold X is $\operatorname{codim} Z = \dim X - \dim Z$

Theorem 1.3.7. *Let $f : X \to Y$ be an embedding, i.e., an injective immersion, which is proper (that is, for every compact in Y, its preimage is compact). Then, the image of f is a submanifold.*

Definition 1.3.8. For a smooth map of manifolds $f : X \to Y$, a point $y \in Y$ is a *regular value* for f if $df_x : T_x(X) \to T_y(Y)$ is exhaustive at every point x such that $f(x) = y$. In this case, $x \in X$ is called a *regular point*. If a point $x \in X$ is not a regular point, it is called a *critical point*. In this case, $y = f(x) \in Y$ is not surjective (exhaustive) and y is called a *critical value*.

Theorem 1.3.9 (Regular Value Theorem)**.** *Let $f : X \to Y$ be a smooth map between smooth manifolds. For every regular value $y \in Y$, $f^{-1}(y)$ is either $\emptyset$ or a submanifold of dimension* $\dim X - \dim Y$ *and* $T_{f^{-1}(y)}X \cong \ker(df)_x$.

Proof. We have to work locally. Fix a given point x in the preimage of y ($x \in f^{-1}(y)$).

By the Local Submersion Theorem (1.3.5), there exist coordinates in some open neighborhoods of x, y such that $f(x_1, ..., x_k) = (x_1, ..., x_l)$ for $l \leq k$ and $y = (0, ..., 0)$. If V is the neighborhood of x. Then $f^{-1}(y) \bigcap V$ is the set of points where $x_1 \cdots = x_l = 0$ because $f(f^{-1}(y)) = 0$ in an enough small neighborhood of x where Theorem 1.3.5 holds true. The functions $x_{l+1}, ..., x_k$ form, therefore, a coordinate system on the set $f^{-1}(y) \bigcap V$ (which is a relatively open subset, what means that is open in the induced topology). So, together, these functions then form a diffeomorphism between the set $f^{-1}(y)$ and an Euclidean space.

We also have, by the fact that y is a regular value, a surjection of tangent spaces from x to y. This ensures the smoothness of the solution set $f^{-1}(y)$, of the inverse function.

Finally, as the set $f^{-1}(y)$ is cut out by these functions $x_{l+1}, ..., x_k$, then

$$\dim(f^{-1}(y)) = k - (l + 1) + 1 = k - l = \dim X - \dim Y \tag{1.3.7}$$

□

1.3.2 Transversality

Definition 1.3.10. A mapping $f : X \to Y$ is said to be *transversal* to the submanifold $Z \subset Y$, and it is denoted by $f \pitchfork Z$, if the *transversality equation* $\operatorname{Im} df_x + T_yZ = T_yY$ holds at every $x = f^{-1}(y) \in f^{-1}(Z)$.

Example 1.3.11. If f is the inclusion map $i : X \to Y$, the transversality equation becomes $T_xX + T_yZ = T_yY$.

Theorem 1.3.12. *If the smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then the preimage $f^{-1}(Z)$ is a submanifold of X. Furthermore, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.*

Example 1.3.13. One particular case of transversality of $f : X \to Y$ with Z is when $Z = \{y\}$, with $y \in Y$. It is the case of the Regular Value Theorem.

If Z is a point $y \in Y$, then $T_yZ = \{0\}$. Thus, transversality holds if and only if $\operatorname{Im} df_x = T_yY$, i.e., if and only if f is regular at $f^{-1}(y)$, where $f : X \to Y$. So transversality includes regularity as a special case.

Remark 1.3.14. If f is the inclusion $i : X \to Y$ and $Z \subset Y$, then $i^{-1}(Z) = X \cap Z$ and $\operatorname{Im} di_x = T_yY$, where $y = f(x)$. The transversality equation is written as $T_xY + T_xZ = T_xY$, which is obviously true, so $X \pitchfork Z$.

Theorem 1.3.15. *The intersection of two transversal submanifolds X and Z of Y is a transversal submanifold. Moreover,* $\operatorname{codim}(X \cap Z) = \operatorname{codim}(X) + \operatorname{codim}(Z)$.

Example 1.3.16. Take $f : t \mapsto (0, t) : \mathbb{R} \to \mathbb{R}^2$ and $Z = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. Then, f is transversal to Z.
Now take $f : t \mapsto (t, t^2) : \mathbb{R} \to \mathbb{R}^2$ and Z again as the x-axis.Then, f is not transversal to Z because at zero the span of the tangent vectors is the same.

Example 1.3.17. The plane xy and the z axis intersect transversally:

$$< (1,0,0),(0,1,0) > + < (0,0,1) >= T_0(xy) + T_0(z) = \tag{1.3.8}$$
$$= T_0(\mathbb{R}^3) =< (1,0,0),(0,1,0),(0,0,1) > \tag{1.3.9}$$

Example 1.3.18. The xy plane and the plane spanned by $< (3,2,0),(0,2,1) >$ are transversal:

As before, we want $T_y(X) + T_y(Z) = T_y(Y)$. That is, the generated space by $T_y(X)$ and by $T_y(Z)$ is indeed $\mathbb{R}^3$ and it suffices that the matrix generated by 3 of these 4 vectors is non-degenerate. Indeed:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1 \neq 0 \tag{1.3.10}$$

Example 1.3.19. The x axis and the plane spanned by $< (1,2,0),(1,1,0) >$ are not transversal:

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0 \tag{1.3.11}$$

Example 1.3.20.

$\overbrace{\mathbb{R}^a \times \{0\}}^{c\ components}$ and $\overbrace{\{0\} \times \mathbb{R}^b}^{c\ components}$ are transversal in $\mathbb{R}^c$ if $a + b \geq c$

If $a + b < c$ it is not true, because we have the following matrix:

$$\begin{vmatrix} 1 & ... & 0 & ... & 0 & ... & 0 & ... & 0 \\ 0 & \ddots & 0 & ... & 0 & ... & 0 & ... & 0 \\ 0 & ... & 1 & ... & 0 & ... & 0 & ... & 0 \\ 0 & ... & 0 & ... & 0 & ... & 1 & ... & 0 \\ 0 & ... & 0 & ... & 0 & ... & 0 & \ddots & 0 \\ 0 & ... & 0 & ... & 0 & ... & 0 & ... & 1 \end{vmatrix} = 0 \tag{1.3.12}$$

Example 1.3.21. Let V be a vector space. $V \times \{\vec{0}\}$ and $\triangle V$ (the diagonal) are transversal in $V \times V$:

$\forall(\vec{u}, \vec{v}) \in V \times V$ as $(\vec{u}, \vec{v}) = (\vec{u}, 0) - (\vec{v}, 0) + (\vec{v}, \vec{v}) = (\vec{u} - \vec{v}, 0) + (\vec{v}, \vec{v})$ and $(\vec{u} - \vec{v}, 0) \in V \times \{\vec{0}\}$ and $(\vec{v}, \vec{v}) \in \triangle V$. Thus the two subspaces intersect transversally (in $0 \vec{\times} 0$).

Example 1.3.22. The subgroup of symmetric matrices ($S(n) = \{A \in M(n) : A^t = A\}$) and skew symmetric matrices ($A^t = -A$) are transversal in $M(n)$:

We can express $\forall C \in M(n)$ as $C = \frac{1}{2}(C + C^t) + \frac{1}{2}(C - C^t)$ and $\frac{1}{2}(C + C^t) \in S(n)$ and $\frac{1}{2}(C - C^t)$ is a skew symmetric matrix. Thus both, subspaces intersect transversally.

1.4 Homotopy and Stability

The idea of *stability* is that something is stable if it remains the same after a small perturbation.

Definition 1.4.1. Let X and Y be two smooth manifolds and $f_0, f_1 : X \to Y$ two smooth functions. An *homotopy* between f_0 and f_1 is a continuous map $F : X \times [0, 1] \to Y$ such that, for every $x \in X$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We also call f_1 a *deformation* of f_0 and denote $F(x, t)$ by $f_t(x)$.

Remark 1.4.2. We will always assume that our homotopies are smooth.

Remark 1.4.3. Homotopy is an equivalence relation on smooth maps from X to Y and the equivalence class is its homotopy class.

Definition 1.4.4. A property is called *stable* if, whenever $f_0 : X \to Y$ possesses the property and $f_t : X \to Y$ is an homotopy of f_0, then, for some $\varepsilon > 0$, and for each $t \in (0, \varepsilon)$, f_t also possesses this property.

Example 1.4.5. Consider curves in the plane., i.e., smooth maps from $\mathbb{R}$ to $\mathbb{R}^2$. The property that a curve passes through the origin is not stable since a small wiggle can immediately distort a smooth curve to avoid 0. The transversal intersection with the x-axis is, on the other hand, a stable property.

Example 1.4.6. The condition $\det A \neq 0$ is stable.

Theorem 1.4.7 (Stability Theorem)**.** *The following classes of smooth maps of a compact manifold X into a manifold Y are stable classes:*

1. *Local diffeomorphisms,*
2. *Immersions,*
3. *Submersions,*
4. *Embeddings,*
5. *Maps transversal to any fixed submanifold $Z \subset Y$,*
6. *Diffeomorphisms.*

Proof. We are going to prove that immersions are a stable class. It automatically implies that local diffeomorphisms are a stable class, because a local diffeomorphism is an immersion in which the target space has the same dimension than the source space.

Assume that $f : X \to Y$ is a smooth immersion and X is compact. Assume that $f_t(x)$ is an homotopy of f. We must produce an $\varepsilon > 0$ such that f_t is an immersion for all $t \in (0, \varepsilon)$. Equivalently, we have to find an $\varepsilon > 0$ such that $d(f_t)_x$ is injective for every $(x, t) \in X \times (0, \varepsilon)$.

By hypothesis, $f_0 = f : X \subset \mathbb{R}^k \to Y \subset \mathbb{R}^l$ (with $k < l$) is an immersion. Then, if we consider $d(f_0)_{x_0}$, with $x_0 \in X$, its injectivity is provided by hypothesis. It implies that the $l \times k$ differential matrix

$$\left(\frac{\partial (f_0)_l}{\partial x_k}(x_0) \right)_{i,j}$$

contains a $k \times k$ submatrix C which is non singular.

The determinant of a matrix is a continuous mapping, and so is each component $\frac{\partial (f_t)_l}{\partial x_k}(x_0)$ of the differential matrix with respect to t and x. Then, since $\det C \neq 0$, for any (x,t) in a neighbourhood U of $(x_0, 0)$ the submatrix C has a non zero determinant. Hence, $d(f_t)_x$ is injective $\forall (x,t)$ in this neighbourhood.

Since X is compact, any open neighbourhood U of $X \times \{0\}$ contains $X \times [0, \varepsilon]$ for some $\varepsilon > 0$. Then, $d(f_t)_x$ is injective in $X \times [0, \varepsilon]$ and we have stability of immersions. □

1.5 Sard's Theorem and Morse Functions

1.5.1 Sard's Theorem

Theorem 1.5.1 (Sard's Theorem, "almost every point" version)**.** *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then, almost every point in Y is a regular value of f. In other words, for almost every $y \in Y$, $df_x : T_xX \to T_yY$ is surjective for any $x \in f^{-1}(y)$.*

We introduce the concept of *zero measure* to restate Sard's Theorem.

Definition 1.5.2. A set $A \subset \mathbb{R}^n$ has *zero measure* if it can be covered by a countable number of regular solids with arbitrary small measure (volume). Namely, for every $\varepsilon > 0$, there exists a countable covering $\{S_1, \dots\}_{i=1}^{\infty}$ such that $A \subset \cup_{i=1}^{\infty} S_i$ and such that $\sum_{i=1}^{\infty} \text{Volume}(S_i) < \varepsilon$.

The concept of zero measure can be extended to manifolds in local parametrizations in the following way.

Definition 1.5.3. Given a manifold X, a set $A \subset X$ has zero measure if $\varphi_\alpha^{-1}(A)$ has zero measure for every parametrization φ_α.

Theorem 1.5.4 (Little Sard's Theorem)**.** *Let $f : X \to Y$ be a differentiable map between smooth manifolds. Suppose* $\dim X < \dim Y$*. Then, $f(X)$ has zero measure.*

Definition 1.5.5. Let $f : X \to Y$ be a smooth map between smooth manifolds. A point $x \in X$ is a *critical point* of f if df_x is not surjective. A point $y \in Y$ is a *critical value* of f if there exists $x \in f^{-1}(y)$ such that x is a critical point.

Theorem 1.5.6 (Sard's Theorem, "zero measure" version)**.** *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then, the set of critical values of f has measure zero.*

Corollary 1.5.7. *The regular values of any smooth map $f : X \to Y$ are dense in Y. In fact, if $f_i : X_i \to Y$ is a countable set of smooth maps, with X_i a countable set of smooth manifolds, the set of points in Y that are simultaneously regular values for all f_i is dense in Y.*

Proof. For any i, consider C_i to be the set of $y \in Y$ such that y is a critical value of f_i. By Sard's Theorem, we know that C_i has zero measure. For any $\varepsilon > 0$ and for each i, consider $\{S_1^i, S_2^i, \dots\}$, a countable covering of C_i ($C_i = \cup_j S_j^i$), such that each S_j^i has zero measure and

$$\sum_{j=1}^{\infty} Volume(S_j^i) < \varepsilon/2^i =: \varepsilon_i.$$

Then, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} Volume(S_j^i) < \sum_{i=1}^{\infty} \varepsilon_i = \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$ □

Remark 1.5.8. It is not true that the set of critical points of a function has measure zero. For instance, take $f : X \to Y$ a constant function and then any point of X is a critical point of f.

Sard's Theorem can be applied to prove that Morse functions are "dense" and to prove the Whitney Embedding Theorem.

1.5.2 Morse Functions

Consider a differentiable manifold M without boundary, and let $f : M \to \mathbb{R}$ be a smooth function.

Definition 1.5.9. A point $p \in M$ is called a *critical point* of the function f if the tangent map $df_p : T_pM \to T\mathbb{R} \cong \mathbb{R}$ is zero. In this case, we say that $f(p)$ is a *critical value*.

Definition 1.5.10. Let p be a critical point of a function $f \in \mathcal{C}^\infty(M)$. The *Hessian* of f at p is the bilinear map

$$\begin{array}{rccc} H_p[f]: & T_pM \times T_pM & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & v(X_u(f)) \end{array}$$

Lemma 1.5.11. *The Hessian $H_p[f]$ is well defined (this means, it does not depend on the choice of X_u), and it is a bilinear and symmetric map.*

Proof. We check that it is symmetric:

$$H_p[f](v,w) - H_p[f](w,v) = v(X_w(f)) - w(X_v(f)) = X_v|_p (X_w(f)) - X_w|_p (X_v(f)) =$$

$$= [X_v, X_w]|_p (f) = d_pf \cdot [X_v, X_w] = 0,$$

where the last term is zero because $d_p f = 0$ because p is a critical point for f. Thus, $H_p[f](v, w) = H_p[f](w, v)$.

Looking again at the definition,

$$H_p[f](v, w) = v(X_w(f)),$$

we see that this does not depend on the extension X_v, as in the expression it only depends on $X_v(p) = v$. On the other hand, $H_p[f](v, w) = H_p[f](w, v)$, so, by the same argument, the Hessian does not depend on the extension chosen for w. This proves that the Hessian is well defined.

Finally, the Hessian is bilinear, because

$$H_p[f](\alpha u + \beta v, w) = (\alpha u + \beta v)(X_w(f)) =$$

$$= \alpha u(X_w(f)) + \beta v(X_w(f)) = \alpha H_p[f](u, w) + \beta H_p[f](v, w),$$

and the same argument applies to the second component by symmetry. □

Remark 1.5.12. If $(x_1, ..., x_n)$ is a local chart centered at a critical point $p \in M$ and $\widetilde{f}$ is the local representation of f in this chart, then the local expression of $H_p[f]$ is the matrix

$$\widetilde{H}_p[f] := \left(\frac{\partial^2 \widetilde{f}}{\partial x_i \partial x_j}(0) \right)_{i,j}.$$

Definition 1.5.13. The *index* of p is the dimension of the maximal subspace $V \subset T_p M$ such that $H_p[f]|_V$ is negative definite.

Definition 1.5.14. A *non-degenerate critical point* of f is a point p such that in any local chart a local representation $H_p[f]$ has maximal rank.

Remark 1.5.15. The definitions of index and non-degenerate critical point are independent of the choice of coordinates, because the index and nullity of a matrix are independent of the basis chosen to represent it, so they are also invariant under any change of coordinates.

Definition 1.5.16. A function $f \in \mathcal{C}^\infty(M)$ is a *Morse function* if all its critical points are non-degenerate. If f is a Morse function, we define

$$\mathrm{Crit}(f) = \{p \in M \; | df_p = 0\},$$

$$\mathrm{Crit}_k(f) = \{p \in \mathrm{Crit}(f) \mid p \text{ has index } k\}.$$

From the Local Submersion Theorem (1.3.5), the neighbourhood of any regular point is regular. For the case of a critical point, we have the Morse Lemma.

Lemma 1.5.17 (Morse Lemma)**.** *Let $p \in \mathrm{Crit}_k(f)$. Then, there is a local coordinate system $(U, (y_1, ..., y_n))$ centered at p such that*

$$f|_U = f(p) - y_1^2 - ... - y_k^2 + y_{k+1}^2 + ... + y_n^2.$$

Proof. A local expression $\widetilde{f}$ of f can be derived from the fundamental theorem of calculus:

$$\widetilde{f}(x) = \widetilde{f}(0) + \int_0^1 \frac{df(tx_1, ..., tx_n)}{dt} dt = f(p) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial \widetilde{f}}{\partial x_i}(tx_1, ..., tx_n) dt.$$

Take $g_i(x) = \displaystyle\int_0^1 \frac{\partial \widetilde{f}}{\partial x_i}(tx_1, ..., tx_n) dt$ and write $\widetilde{f}$ as

$$\widetilde{f}(x) = f(p) + \sum_{i=1}^n x_i g_i(x).$$

Since $g_i(0) = \frac{\partial \widetilde{f}}{\partial x_i}(0) = 0$, we can apply the same procedure for each i, so there are functions h_{ij} such that

$$g_i(x) = \sum_{j=1}^n x_j h_{ij}(x),$$

$$\widetilde{f}(x) = f(p) + \sum_{i,j=0}^n x_i x_j h_{ij}(x).$$

These functions satisfy that

$$h_{ij}(0) = \frac{1}{2} \frac{\partial^2 \widetilde{f}}{\partial x_i \partial x_j}(0),$$

and

$$h_{ij} = h_{ji}.$$

Then, we apply inductively a change of coordinates following, at each step, the next idea:

Suppose that there is a local coordinate system $(U_1, (u_1, ..., u_n))$ (with $U_1 \subseteq U$) such that

$$f = f(p) \pm u_1^2 \pm ... \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_i H_{ij}(u),$$

where $(H_{ij})_{i,j}$ form a symmetric matrix and $(H_{ij}(0))_{i,j}$ form a non-degenerate matrix. Let us suppose that $H_{rr}(0) \neq 0$ (if it is not the case, we can apply a linear change of coordinates to ensure it). Take $S(u) = \sqrt{|H_{rr}(u)|}$, which will be a non-vanishing positive function of u in a neighbourhood $U_2 \subset U_1$ of 0. Thus, we can introduce the new local coordinates $(v_1, ..., v_n)$ on U_2 as

$$v_i = u_i \quad \text{for } i \neq r,$$

$$v_r(u) = S(u) \left[u_r + \sum_{i>r} u_i \frac{H_{ir}(u)}{H_{rr}(u)} \right].$$

Using the inverse function theorem we conclude that $(v_1, ..., v_r)$ form an invertible smooth set of coordinates in a neighbourhood of the origin, $U_3 \subset U_2$. It can also be seen that

$$f(v) = f(p) + \sum_{i \leq r} \left(\pm v_i^2\right) + \sum_{i,j>r} v_i v_j G_{ij}(v),$$

where G_{ij} are symmetric and form a non-degenerate matrix at $v = 0$.

Therefore, after applying these steps n times we can construct the coordinate system in some neighbourhood U of p satisfying the claimed properties. □

Corollary 1.5.18. *The set of non-degenerate critical points of a differentiable function is isolated.*

Example 1.5.19. The height function of a Torus is a Morse function.

Theorem 1.5.20. *Let $X \hookrightarrow \mathbb{R}^N$ be a smooth manifold and $f : X \to \mathbb{R}$ a smooth function. Then, for almost all $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $f_a(x) := f(x) + \langle a, x \rangle = f(x) + a_1 x_1 + \cdots + a_N x_N$ is a Morse function.*

This theorem comes to say that Morse functions are "dense". To prove it, we start proving a lemma.

Lemma 1.5.21. *Let $f : U \subset \mathbb{R}^k \to \mathbb{R}$ be a smooth function on an open set U. Then, for almost all k-tuples $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, the function $f_a := f + a_1 x_1 + \cdots + a_k x_k$ is a Morse function on U.*

Proof. Consider $g : U \to \mathbb{R}^k$ defined as $g(x_1, \ldots, x_k) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k} \right)$. Observe that $(df_a)_p = g(p) + (a_1, \ldots, a_k)$. Then p is a critical point of f if and only if $g(p) + (a_1, \ldots, a_k) = 0$. So, if $g(p) = -(a_1, \ldots, a_k) = -a$, p is a critical point of f_a.

Notice that f and f_a have the same second derivatives, in both cases, the Hessian is dg_p. The critical point p is non-degenerate if dg_p is non-singular.

Now, assume $-a$ is a regular value for g. Then, for any p such that $g(p) = -a$, the differential dg_p is non-singular and, hence, p, the critical point of f_a is non-degenerate.

Applying Sard's Theorem (1.5.1) to the function $g : U \to \mathbb{R}^k$, which is a smooth map between smooth manifolds, we conclude that almost every point in $\mathbb{R}^k$ is a regular value of g. In other words, for almost every $a \in \mathbb{R}^k$, $dg_x : \mathbb{R}^k \to \mathbb{R}^k$ is surjective for any $x \in g^{-1}(a)$. We conclude that for almost every $a \in \mathbb{R}^k$, the Hessian $H(f_a) = dg$ at every critical point of f is non-singular, so f is a Morse function. □

Now, we prove Theorem 1.5.20 on denseness of Morse functions on any smooth manifold.

Proof of Theorem 1.5.20. We have $f : X \to \mathbb{R}$ a smooth function, with $X \subset \mathbb{R}^N$ and, then, $k := \dim X \leq N$. We want to prove that $f_a = f + a_1x_1 + \cdots a_Nx_N$ is a Morse function for almost all $a = (a_1, \dots, a_N) \in \mathbb{R}^N$.

Suppose $x \in X$ and set $x = (x_1, \dots, x_N)$, where x_i are the standard coordinate functions on $\mathbb{R}^N$ (considering $\varphi^{-1}(x)$ the parametrization, i.e. $(\varphi_1, \dots, \varphi_N)$ the coordinate functions). Since the dimension of X is k, we know that We know that the differential at any point $x \in X$ spans a vector space of dimension k. In other words, $\operatorname{rank}(d(x_1, \dots, x_N)) = k$, where $d(x_1, \dots, x_N)$ is the differential of the coordinate function. Then, there exist $(x_{i_1}, \dots, x_{i_k}) : X \to \mathbb{R}^k$ that define a local diffeomorphism when restricted to X. Assume that $df(x_1, \dots, x_k)$ is non-singular and, so, $(x_1, \dots, x_k)$ is a coordinate system.

$$\begin{array}{rccll} \varphi^{-1}: & \mathbb{R}^k & \longrightarrow & U \subset X & \\ & p & \longmapsto & \varphi^{-1}(p) & = (x_1, \dots, x_k) \end{array}$$

We can cover X with open subsets U_α such that, on each, some k of the coordinate functions form a coordinate system. By the second axiom of countability, we can assume that there are countable many U_α.

Let $X = \cup \varphi_\alpha(U_\alpha)$, with $U_\alpha \subset \mathbb{R}^k$. Take $c = (c_{k+1}, \dots, c_N)$ and consider the function $f_{(0,c)} : U_\alpha \to \mathbb{R}$ defined as $f_{(0,c)} = f + \underbrace{c_{k+1}x_{k+1} + \cdots + c_Nx_N}_{fixed}$.

We apply Lemma 1.5.21 (which says the same as the theorem we are proving, but only for open sets) to $f_{(0,c)}$. The Lemma implies that, for almost all $b = (b_1 \dots, b_k) \in \mathbb{R}^k$, the new function $f_{(b,c)} := f_{(0,c)} + b_1x_1 + \cdots + b_kx_k$ is a

Morse function on U_α.

Set $S_\alpha = \{a \in \mathbb{R}^N \mid f_a \text{ is not a Morse function}\}$. We want to prove that S_α has zero measure. Set $a = (a_1, \dots, a_N) := (b_1, \dots, b_k, c_{k+1}, \dots, c_N)$ and consider $\mathbb{R}^N$ "sliced" by the spaces $\{a_{k+1} = c_{k+1}, \dots, a_N = c_N\}$. The idea of fixing these $N-k$ coordinates is that we make S_α intersect with $\mathbb{R}^k \times \{c\}$ and, for any c, each slice $S_\alpha \cap (\mathbb{R}^k \times \{c\})$ has zero measure by the Lemma 1.5.21 applied on $f_{(0,c)}$.

By Fubini's theorem, since each slice has zero measure, the total measure of S_α is zero in $\mathbb{R}^n$. Since X is second countable, $X = \cup_\Lambda \varphi_\alpha(U_\alpha)$, where Λ is countable. So $S = \cup S_\alpha$ is a countable union of zero measure sets and then S has zero measure. □

Now we restate the Whitney Theorem (1.1.3).

Theorem 1.5.22 (Whitney's Embedding Theorem)**.** *For any compact smooth manifold X of dimension k, there exists an embedding $f : X \to \mathbb{R}^{2k+1}$.*

Proof. We want to prove that every k-dimensional compact manifold $X \subset \mathbb{R}^N$ admits an embedding on $\mathbb{R}^{2k+1}$. In fact, if $X \subset \mathbb{R}^N$ is k-dimensional and $N > 2k + 1$, we will produce a linear projections that restricts to a one-to-one immersion on R^{2k+1} that restricts to a one-to-one immersion of X.

Proceeding inductively, we prove that if $f : X \to \mathbb{R}^M$ is an injective immersion with $M > 2k + 1$ then there exists a unit vector such that the composition of I with the projection map to the orthogonal complement of this vector is still an injective immersion.

We look for a one-to-one immersion of X, which we already know that will be proper because X is compact by hypothesis and for compact manifolds one-to-one immersions are embeddings.

Define the following maps:

$$\begin{array}{rccc} g: & T(X) & \longrightarrow & \mathbb{R}^M \\ & (x,u) & \longmapsto & df_x(u) \end{array}$$

$$\begin{array}{rccc} h: & X \times X \times \mathbb{R} & \longrightarrow & \mathbb{R}^M \\ & (x,y,t) & \longmapsto & t(f(x) - f(y)) \end{array}$$

Assume $M > 2k+1$ (if $M \leq 2k+1$, we are done). By Little Sard's Theorem (1.5.4), there exists an $a \in \mathbb{R}^M$ belonging to neither the image of g or h. Let π be the projection of $\mathbb{R}^M$ onto $H := \langle a \rangle^\perp$. Consider $\pi \circ f : X \to H \simeq \mathbb{R}^{M-1}$, where f denotes the immersion. Let us check that $\pi \circ f$ is an immersion.

- $\pi \circ f$ is injective: $\pi(f(x)) = \pi(f(y)) \Rightarrow f(x) - f(y) = t \cdot a$ for some scalar t. If $x \neq y$, then $t \neq 0$, since f is injective (it is an embedding). But then, $h(x, y, 1/t) = a$, but a is not in the image of h, which is a contradiction.

- $d(\pi \circ f)$ is injective: $d\pi_{f(x)} \circ df_x = \pi \circ df_x$. Observe that $(d(\pi \circ f)_x)(v) = 0$ iff $df_x(v) = t \cdot a$ for some scalar t. Hence, since d is linear, $df_x(v/t) = a$, contradicting the fact that a does not belong to the image of g.

□

1.6 Partitions of Unity

The idea of partitions of unity is to go from local aspects to global aspects on a manifold.

Definition 1.6.1. A *partition of unity* in a manifold X is a set $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in \Lambda}$ such that:

1. For any $x \in X$, there exists an open neighbourhood U such that the number of f_α that satisfy $f_\alpha(U) \neq 0$ is finite.
2. $0 \leq f_\alpha \leq 1$ for any α.
3. For all $x \in X$, $\sum_{\alpha \in \Lambda} f_\alpha(x) = 1$.

Proposition 1.6.2. *Let X be a manifold. Then, there exists a partition of unity subordinated to any atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$. I.e. the support of each f_α is contained in U_α.*

Proof. Without loss of generality, we can assume that $\varphi_\alpha(U_\alpha) \subset B_2(0)$, that $X = \cup_{\alpha \in \Lambda} \varphi_\alpha^{-1}(B_1(0))$ and that the atlas is locally finite (for any $x \in X$, there exists a finite number of charts U_α that contain x).

Take a function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that:

- $f(B_1(0)) = 1$,
- $f(\mathbb{R}^n \setminus B_2(0)) = 0$,
- $0 \leq f \leq 1$,

the so-called *bump function*. This function defines a partition of unity on X. Let us see that it exists and how the partition of unity is defined.

Take the following real function:

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

which is a smooth function. Define a new function

$$h(x) = \frac{g(x)}{g(x) + g(1-x)}$$

that satisfies that $h(x) = 1$ if $x \geq 1$ and $h(x) = 0$ if $x \leq 0$. Now, take $k(x) = h(x+2) \cdot h(-x+2)$ and finally consider $f = k(\|x\|)$, which satisfies the properties of a bump function.

Take the charts

$$\theta_\alpha = \begin{cases} f \circ \varphi_\alpha & \text{in } U_\alpha \\ 0 & \text{else} \end{cases}$$

Then, $\rho_\alpha = \frac{\theta_\alpha}{\sum_\alpha \theta_\alpha}$ adds up to 1 and is a partition of unity. □

1.6.1 Applications of Partition of Unity

The partition of unity is used, for instance, to show that a manifold can be equipped with a *Riemannian metric.*

Definition 1.6.3 (Riemannian metric)**.** A *Riemannian metric* on a manifold X is a map $g_p : T_pX \times T_pX \to \mathbb{R}$ such that it is:

- Bilinear,
- Symmetric,
- Positive definite.

Proposition 1.6.4. *Every manifold admits a Riemannian metric.*

Proof. Take an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of the manifold X. For every $x \in U_\alpha$, define

$$\widetilde{g}_x^\alpha(X_x, Y_x) := \sum_{i=1}^n X_x^i Y_x^i,$$

where

$$X_x = \sum_{i=1}^n X_x^i \frac{\partial}{\partial x_i}, \quad Y_x = \sum_{i=1}^n Y_x^i \frac{\partial}{\partial x_i}.$$

For every $x \in X$, take

$$\bar{g}_x(X_x, Y_x) := \begin{cases} \rho_\alpha \widetilde{g}_x^\alpha(X_x, Y_x) & \text{in } U_\alpha \\ 0 & \text{outside} \end{cases}$$

where ρ_α is a partition of unity.

Finally, take

$$g_x(X_x, Y_x) := \sum_\alpha \bar{g}_x(X_x, Y_x),$$

which is indeed a Riemannian metric. □

1.6.2 Morse Theory and Applications to Topology

The goal of Morse theory is to reconstruct the topology of a manifold from a Morse function. If X is a smooth manifold and $f : X \to \mathbb{R}$ a Morse function, f can give information on the topology of X. An example of usage of this idea is the Reeb Theorem.

Theorem 1.6.5 (Reeb)**.** *If a compact manifold X of dimension n admits a function $f : X \to \mathbb{R}$ which is Morse and has only two critical points, then X is diffeomorphic to S^n ($X \cong S^n$).*

In this section, we denote by X_c the set of $x \in X$ such that $f(x) \leq c$ for a fixed Morse function $f : X \to \mathbb{R}$.

Proposition 1.6.6. *Consider $f : X \to \mathbb{R}$. Take $a, b \in \mathbb{R}$ such that $f^{-1}([a, b]) \subset X$ is compact and does not contain any critical point. Then, X_a is a deformation retract of X_b and, moreover, $X_a \cong X_b$.*

Proof. Let (X, g) be the manifold X with Riemannian metric g, that is, a Riemannian manifold. Then, define the gradient of f as the only solution of $g(\text{grad}, v) = df(v)$ for every vector v. Its uniqueness is due to non-degeneracy of g when seen as $g(\text{grad}, \cdot) = df$.

Let $W \subset X$ be the set of non-critical points of f. Consider any Riemannian metric g on X and take Y to be $Y = \operatorname{grad} f / \| \operatorname{grad} f \|^2 \in \mathfrak{X}(X)$ in W.

Let γ be the maximal integral curve of Y and compute $d/dt(f(\gamma(t))$.

$$\begin{aligned}\frac{d}{dt}(f(\gamma(t)) &= df(\gamma(t)) \cdot \gamma'(t) = \\ &= df(\gamma(t)) \cdot Y(\gamma(t)) = \\ &= g(\operatorname{grad} f(\gamma(t)), Y(\gamma(t))) = \\ &= \frac{1}{\| \operatorname{grad} f\|^2} \cdot g(\operatorname{grad} f, \operatorname{grad} f) \\ &= 1\end{aligned}$$

Then, $f(\gamma(t)) = f(\gamma(0)) + t$.

Remark 1.6.7. $K := f^{-1}([a, b])$ is compact by hypothesis. Take $\gamma(0) \in= f^{-1}(a)$. There are two possible cases:

1. $\gamma(t) \in K$ for any $t \in I, t > 0$. Since K is compact, the flow is defined for all t. Then, $[0, \infty) \subset I$. In particular, $[0, b - a] \subset I$.

2. The solution $\gamma(t)$ goes out of K for some $s > 0$. Then, $b < f(\gamma(s)) = f(\gamma(0)) + s = a + s$, implying that $s > b - a$ and that $[0, b - a] \subset I$.

In both cases, we have that $[0, b - a] \subset I$.

Now, we extend the field Y defined on W to the whole manifold X. Consider a bump function $\psi : X \to \mathbb{R}$ such that $\psi|_K = 1$ and its support is in W. The field $\widetilde{Y}$ defined as

$$\widetilde{Y} := \begin{cases} \psi \cdot Y & \text{in } W \\ 0 & \text{in } X \setminus W = \operatorname{Crit}(f) \end{cases}$$

is well-defined. Let φ^t be the flow of $\widetilde{Y}$ and consider $t = b - a$. The map φ^{b-a} is well defined and it is a diffeomorphism that takes X_a to X_b. Now, we construct the desired retraction $r : X_b \times [0, 1] \to X_b$ in the following way:

$$r(x, t) := \begin{cases} x & \text{if } f(x) \leq a \\ \varphi^{t(a-f(x))}(x) & \text{if } a < f(x) \leq b \end{cases}$$

□

The Reeb Theorem (1.6.5) is a corollary of this proposition.

Proof. Since X is compact, $f(X)$ is compact and we can normalize it without loss of generality and assume $f(X) = [0, 1]$. The two only critical points of f have to be a maximum and a minimum, so there exist $p \in X$ such that $f(p) = 0$

and $q \in X$ such that $f(q) = 1$.

By the Morse Lemma (1.5.17), in a neighbourhood of the minimum critical point p, we can write, $f = x_1^2 + \cdots + x_n^2$, while in a neighbourhood of the maximum q, we can write $f = 1 - x_1^2 - \cdots - x_n^2$. So, for a small $\varepsilon > 0$, the set $\{x \in X \mid x_1^2 + \cdots + x_n^2 < \varepsilon\}$ is a disk D^n. Hence, there exists $\varepsilon > 0$ such that $f^{-1}([0, \varepsilon])$ is diffeomorphic to a disk. The same reasoning at the maximum q leads to the conclusion that $f^{-1}([1-\varepsilon, 1])$ is diffeomorphic to a disk D^n.

Because of the proposition, since there are no more critical points, the sets X_ε and $X_{1-\varepsilon}$ are diffeomorphic. Then, their boundaries $\partial f^{-1}([0,\varepsilon])$ and $\partial f^{-1}([1-\varepsilon, 1])$ are diffeomorphic via φ, the flow of the gradient vector field.

Finally, we want to see that we can gluing the two disks to give the sphere S^n, and we can do it explicitly. S^n is the gluing of two n-disks via the identity map on its boundary (which is $S^{n-1} = \partial D^n$), i.e., $S^n = D_1^n \sqcup_{id} D_2^n / \sim$ with the equivalence relation $\sim$ given by

$$x \in D_1^n \sim y \in D_2^n \quad \iff \quad y = id(x).$$

But X is the gluing of two n-disks glued via φ, not the identity, but this is not a problem, since we can construct $S^n = D_1^n \sqcup_\varphi D_2^n / \sim$ using the equivalence relation:

$$x \in D_1^n \sim y \in D_2^n \quad \iff \quad y = \varphi(x).$$

Then, construct the following diffeomorphism h between the two disks that effectively glues them as is in the sphere case:

$$\begin{array}{rccc} h: & S^n = D_1^n \sqcup_\varphi D_2^n & \longrightarrow & D_1^n \cup_{\text{id}} D_2^n \\ & z & \longmapsto & \begin{cases} \|z\| \cdot \varphi\left(\frac{z}{\|z\|}\right) & \text{if } z \in D_1, z \in D_2 \setminus \{0\} \\ 0 & \text{else} \end{cases} \end{array}$$

□

Theorem 1.6.8. *Let $p \in X$ be a non-degenerate critical point of $f : X \to \mathbb{R}$ and let k be its index. Define $c := f(p)$. Consider $\varepsilon > 0$ small enough such that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and does not contain any other critical points. Then, $M_{c+\varepsilon} = M_{c-\varepsilon} \cup D^k$. It implies that $M_{c+\varepsilon}$ is homotopically equivalent to $M_{c-\varepsilon}$ with a cell attached.*

Chapter 2

Lie Theory

This chapter follows [Bry95] and [DK00].

2.1 Lie Groups and Algebras

Definition 2.1.1 (Lie group)**.** A *Lie group* G is a smooth manifold G that has a group structure and whose maps

$$\begin{array}{rccc} m: & G \times G & \longrightarrow & G \\ & (a,b) & \longmapsto & a \cdot b \end{array}$$

$$\begin{array}{rccc} i: & G & \longrightarrow & G \\ & a & \longmapsto & a^{-1} \end{array}$$

are smooth.

Some examples of Lie groups are $\mathbb{R}$ and $\mathbb{C}$ equipped with the addition operation and $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$ equipped with the matrix product. A classical set of Lie groups is the family of *matrix Lie groups*.

Definition 2.1.2 (Matrix Lie group)**.** A *matrix Lie group* is a subgroup G of $\mathbf{GL}(n, \mathbb{C})$ that is closed in $\mathbf{GL}(n, \mathbb{C})$, meaning that if $(A_m)_{m \in \mathbb{N}} \subset G$ with $\lim_{m \to \infty} A_m = A \in \mathrm{Mat}(n \times n, \mathbb{C})$, then either $A \in G$ or A is not invertible.

The most studied Lie groups are, in fact, matrix Lie groups. Some examples of matrix Lie groups are:

- $SL(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \det A = 1\}$,
- $O(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^t A = I\}$,

- $SO(n,\mathbb{R}) = \{A \in \mathbf{GL}(n,\mathbb{R}) \mid A^tA = I, \det A = 1\}$,
- $U(n) = \{A \in \mathbf{GL}(n,\mathbb{C}) \mid A^*A = I\}$,
- $SU(n) = \{A \in \mathbf{GL}(n,\mathbb{C}) \mid A^*A = I, \det A = 1\}$
- $Sp(2n,\mathbb{R}) = \{A \in \mathbf{GL}(2n,\mathbb{R}) \mid A^tJA = J\}$, with J a nonsingular skew-symmetric matrix (the group of symplectic matrices).

The unit 1-sphere $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ is also a Lie group, where multiplication is given by $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ and inverse by $(e^{i\theta})^{-1} = e^{-i\theta}$. The 3-sphere also admits a Lie group structure with quaternions, $S^3 = \{q \in \mathbb{H} \mid \|q\| = 1\}$, where $\mathbb{H} = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$. S^2 does not admit a Lie group structure, and this is consequence of the Hairy Ball Theorem.

Theorem 2.1.3 (Hairy Ball Theorem)**.** *There does not exist a non-vanishing continuous vector field tangent to the sphere S^n if n is odd.*

Theorem 2.1.4 (Cartan's Closed Subgroup Theorem)**.** *If H is a closed subgroup of a Lie group G, then H is an embedded Lie subgroup.*

Definition 2.1.5. If G is a Lie group, the *left translation* by an element $g \in G$ is defined as:

$$\begin{array}{cccccc} L_g: & G & \longrightarrow & G & \\ & h & \longmapsto & g\cdot h & = m(g,h) \end{array}$$

Similarly, the *right translation* by $g \in G$ is defined as:

$$\begin{array}{cccccc} R_g: & G & \longrightarrow & G & \\ & h & \longmapsto & h\cdot g & = m(h,g) \end{array}$$

Lemma 2.1.6. *L_g and R_g are diffeomorphisms.*

Proof. From the definition, one can check that:

- L_g is smooth.
- $L_{g\cdot g'} = L_g \cdot L_{g'}$.
- $(L_g)^{-1} = L_{g^{-1}}$.

The same for R_g. □

The left and right translations induce the maps $d(L_g)_h : T_hG \to T_{gh}G$ and $d(R_g)_h : T_hG \to T_{hg}G$.

Definition 2.1.7. A vector field $X \in \mathfrak{X}(G)$ *left-invariant* if $d(L_g)_h X_h = X_{gh}$. We denote by $\mathfrak{X}(G)^{L(G)}$ the set of left-invariant vector fields. In the same way we can define *right-invariant* vector fields.

Definition 2.1.8. Let $e \in G$ denote the identity element of the Lie group G. We denote the *tangent space* of G at e, T_eG, by $\mathfrak{g}$.

Lemma 2.1.9. *There is a one-to-one correspondence between $\mathfrak{X}(G)^{L(G)}$ and $\mathfrak{g}$.*

Proof. Take $X \in \mathfrak{g}$ and take $\widetilde{X}_g = d(L_g)_e X \in T_gG$, where $d(L_g)_e : T_eG = \mathfrak{g} \to T_gG$. Then, it is clear that $\widetilde{X} \in \mathfrak{X}(G)$. Since $m : G \times G \to G$ is smooth, so is $m(g, \cdot) = L_g$. Now

$$\begin{array}{rccll} X: & G & \longrightarrow & T_gG & \\ & g & \longmapsto & \widetilde{X}_g & = d(L_g)_e X \end{array}$$

since X is a section such that $\pi \circ X = Id_G$,with π the projection $\pi : TG \to G$. Moreover, $\widetilde{X}_g$ is a left-invariant vector field, i.e., $d(L_g)_h \widetilde{X}_h = \widetilde{X}_{gh}$. We prove it:

$$\begin{aligned} d(L_g)_h \widetilde{X}_h &= d(L_g)_h d(L_h)_e X_e = \\ &= d(L_g \circ L_h)_e X_e = \\ &= d(L_{gh})_e X_e = \\ &= \widetilde{X}_{gh} \end{aligned}$$

Finally, since L_g is a diffeomorphism, $d(L_g)_e$ is a local isomorphism, so we get the one-to-one correspondence. □

Back to S^2. Suppose that S^2 has a Lie group structure and find a contradiction. Take $X \in T_eS^2$, $X \neq 0$. Then, obtain $\widetilde{X}$ a left-invariant vector field on S^2. Since $d(L_g)_e$ is a local diffeomorphism, $\widetilde{X}$ is nowhere zero.

Actually, $\mathfrak{g}$ is more than a vector space. Endowed with a Lie bracket, it is a Lie algebra. We recall its definition.

Definition 2.1.10 (Lie Algebra)**.** The pair $(\mathfrak{g}, [\cdot,\cdot])$ is a *Lie algebra* if $\mathfrak{g}$ is a vector space and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear, skew-symmetric and satisfies Jacobi identity.

The definition of Lie algebra arises from the fact that each Lie group has an associated Lie algebra.

Definition 2.1.11 (Lie Algebra of a Lie Group)**.** Let G be a Lie group, and consider $\mathfrak{g} = T_eG$. Consider $[\cdot,\cdot]$ defined over $\mathfrak{g}$ as $[X, Y] := [d(L_g)_e X, d(L_g)_e Y] = [\widetilde{X}, \widetilde{Y}]_e$. $(\mathfrak{g}, [\cdot,\cdot])$ is called the *Lie algebra associated to the Lie group G.*

Remark 2.1.12. $\dim \mathfrak{g} = \dim G$.

2.1.1 Lie Algebras associated to Lie Groups

The general procedure to compute the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ of a Lie group G is:

1. Find $e \in G$.
2. Find $\mathfrak{g} = T_e G$.
3. Compute $\widetilde{X}_g = d(L_g)_e X$ for an $X \in \mathfrak{g}$.
4. Compute $[X, Y] = [\widetilde{X}, \widetilde{Y}]_e$.

Example 2.1.13. The Lie group $G = (\mathbb{R}^n, +)$, with $e = 0$ and $\mathfrak{g} = T_0\mathbb{R}^n = \mathbb{R}^n$. We compute the Lie Bracket in the following way. Take $X \in \mathfrak{g} = \mathbb{R}^n$ and compute $\widetilde{X}_g = d(L_g)_0 X$. To compute $d(L_g)_0 X$, we will use the fact that for any map f, any point p and any vector v, the following equality is satisfied

$$df_p v = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}$$

if $\gamma(0) = p$ and $\gamma'(0) = v$.

Define $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ such that $\gamma(0) = e = 0 \in \mathbb{R}^n$ and $\gamma'(0) = X$. The natural γ that satisfies both conditions is $\gamma(t) = 0 + tX$. Now, $d(L_g)_0 X = \frac{d}{dt}(L_g \circ \gamma(t))|_{t=0} = \frac{d}{dt}(g + (0 + tX))|_{t=0} = X|_{t=0} = X$. Then, $\widetilde{X}_g = X$ and $\widetilde{Y}_g = Y$ are both constant vector fields. Therefore, $[\widetilde{X}, \widetilde{Y}]_0 = [X, Y]$ constant and, hence, $[X, Y] = \mathcal{L}_X Y = 0$.

The Lie algebra associated to $G = (\mathbb{R}^n, +)$ is:

$$(\mathbb{R}^n, [X, Y] = 0).$$

Example 2.1.14. The Lie group $G = (\mathbf{GL}(n, \mathbb{R}), \cdot)$, with $e = I_n$ and $\mathfrak{g} = T_{I_n} G = \mathrm{Mat}(n \times n, \mathbb{R})$. Take $X \in \mathfrak{g}$ and compute $\widetilde{X}_g = d(L_g)_{I_n} X$ using the same trick of the previous example.

Define $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{GL}(n, \mathbb{R})$ such that $\gamma(0) = e = I_n$ and $\gamma'(0) = X$. The natural γ that satisfies both conditions is $\gamma(t) = I_n + tX$. Now, $d(L_g)_{I_n} X = \frac{d}{dt}(L_g \circ \gamma(t))|_{t=0} = \frac{d}{dt}(g \cdot (I_n + tX)|_{t=0} = gX|_{t=0} = gX$. Then, $\widetilde{X}_g = gX$ and $\widetilde{Y}_g = gY$, so they are non-constant vector fields.

Now, defining $\gamma^1(t) = I_n + tX$ and $\gamma^2(t) = I_n + tY$, we compute:

$$\begin{aligned}
[\widetilde{X}, \widetilde{Y}]_{I_n} &= \widetilde{X}_{I_n}(\widetilde{Y}) - \widetilde{Y}_{I_n}(\widetilde{X}) = \\
&= \frac{d}{dt}(\widetilde{Y}_{I_n} \circ \gamma^1(t))|_{t=0} - \frac{d}{dt}(\widetilde{X}_{I_n} \circ \gamma^2(t))|_{t=0} = \\
&= \frac{d}{dt}((I_n + tX)I_n Y)|_{t=0} - \frac{d}{dt}((I_n + tY)I_n X)|_{t=0} = \\
&= XY|_{t=0} - YX)|_{t=0} = \\
&= XY - YX
\end{aligned}$$

which is known as the commutator of matrices.

The Lie algebra associated to $G = (\mathbf{GL}(n, \mathbb{R}), \cdot)$ is:

$$(\mathrm{Mat}(n \times n, \mathbb{R}), [X, Y] = XY - YX).$$

2.1.2 Lie Groups and Lie Algebras Homomorphisms

Definition 2.1.15. Let G, H be Lie groups. A *Lie group homomorphism* $\phi : G \to H$ is a morphism between groups and a smooth map between manifolds.

Remark 2.1.16. It is sufficient to ask ϕ to be continuous to obtain smoothness.

Definition 2.1.17. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be Lie algebras. A *Lie algebra homomorphism* ϕ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$[\phi(X), \phi(Y)]_{\mathfrak{h}} = \phi([X, Y]_{\mathfrak{g}})$$

Lemma 2.1.18. *Let $\phi : G \to H$ be a Lie group homomorphism. Then, $d\phi_e : T_{e_G}G = \mathfrak{g} \to T_{e_H}H = \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. We have to check that ϕ is a Lie algebra homomorphism, i.e., that it is a linear map such that $[d\phi_e(X), d\phi_e(Y)]_{\mathfrak{h}} = d\phi_e([X, Y]_{\mathfrak{g}})$.

$$\begin{aligned}
d\phi_e([X, Y]_{\mathfrak{g}}) &= d\phi_e([d(L_g)_e X, d(L_g)_e Y]_{\mathfrak{g}}) = \\
&= [d(L_h)_e d\phi_e X, d(L_h)_e d\phi_e Y]_{\mathfrak{h}} = \\
&= [d\phi_e X, d\phi_e Y]_{\mathfrak{h}}
\end{aligned}$$

Now, the left hand side is equal to $d\phi_e([\widetilde{X}, \widetilde{Y}]_{\mathfrak{g}})$ and the right hand side is equal to $[\widetilde{d\phi_e X}, \widetilde{d\phi_e Y}]_{\mathfrak{h}}$.

We claim that $\widetilde{d\phi_e X_h} = d\phi_e \widetilde{X}_g$. Indeed:

$$d\phi_e \widetilde{X}_g = d\phi_e(d(L_g)_e X) = d(\phi \circ L_g)_e X = d(L_h \circ \phi)_e X = \widetilde{d\phi_e X_h}.$$

Using this claim, we get

$$d\phi_e([\widetilde{X}, \widetilde{Y}]_{\mathfrak{g}}) = [d\phi_e \widetilde{X}, d\phi_e \widetilde{Y}]_{\mathfrak{h}}$$

□

Corollary 2.1.19. *The Lie algebras of two isomorphic Lie groups are isomorphic.*

Lemma 2.1.20. *Let G be a matrix Lie group. Then, the Lie bracket of G is the usual Lie bracket of matrices, i.e., the Lie bracket of the Lie group $\mathbf{GL}(n, \mathbb{C})$.*

Proof. To prove the lemma, just consider the inclusion map $i : G \to \mathbf{GL}(n, \mathbb{R})$. □

The next examples show what are the Lie algebras of some classical matrix Lie groups.

Example 2.1.21. Take $O(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^t A = I\}$. Then, $\mathfrak{o}(n) := T_{I_n} O(n, \mathbb{R})$. Let $X \in \mathfrak{o}(n)$, so X is the representative of the paths γ such that $\gamma(0) = I_n$ and $\dot{\gamma}(0) = X$. One obvious candidate is $\gamma(t) = I_n + tX$. But in order to guarantee that $\gamma(t) \in O(n, \mathbb{R})$ for every $t \in (-\varepsilon, \varepsilon)$ we consider $\gamma(t) = I_n + tX + O(t^2)$. Then, the path is really on $O(n, \mathbb{R})$ if and only if $\gamma(t)^t \gamma(t) = I_n$. Then, it is necessary that

$$(I_n + tX + O(t^2))^t \cdot (I_n + tX + O(t^2)) = I_n + t(X + X^t) + O(t^2) = I_n.$$

Then, $X + X^t = 0$. The other terms in $O(t^2)$ do not play any role. Hence, $\mathfrak{o}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A + A^t = 0\}$.

Example 2.1.22. Take $SO(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^t A = I, \det A = 1\}$. Then, by the same computations done for $O(n, \mathbb{R})$, $\mathfrak{so}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A + A^t = 0\}$.

Example 2.1.23. Take $SL(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \det A = 1\}$. Let $X \in \mathfrak{sl}(n)$, so X is the representative of the paths $\gamma(t) \in SL(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\dot{\gamma}(0) = X$. For any of such $\gamma(t)$, we have that $\det(\gamma(t)) = 1$. Then, if $\gamma(t) = (a_{i,j}(t))$, we know that: $a_{i,j}(0) = \delta_{i,j}$. Besides, we know that

$$\det(\gamma(t)) = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}(t) \right) = 1.$$

Hence, deriving with respect to t, we obtain that

$$\sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \left(\prod_{i=1}^{n} a_{i,\sigma_i}(t) \right) \left(\sum_{i=1}^{n} \frac{a'_{i,\sigma_i}(t)}{a_{i,\sigma_i}(t)} \right) \right) = 0.$$

Evaluating the latter expression at $t = 0$, we deduce that:

$$\sum_{i=1}^{n} a'_{i,i}(0) = 0.$$

Then, $\mathfrak{sl}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \mathrm{tr}\, A = 0\}$.

Definition 2.1.24. A smooth manifold M is *parallelizable* if there exist $X_1, \ldots, X_n \in \mathfrak{X}(M)$ such that, for every $p \in M$ the set $\{(X_1)_p, \ldots, (X_n)_p\}$ forms a basis of T_pM.

Lemma 2.1.25. *Every Lie group G is parallelizable*

Proof. If G is a Lie group, then any basis of $\mathfrak{g} = T_eG$ is a smooth global frame for G. □

Remark 2.1.26. A non-orientable manifold does not admit a Lie group structure.

Lemma 2.1.27. *Let $f : M \to N$ be a diffeomorphism and $X, Y \in \mathfrak{X}(M)$. Then, $df([X, Y]_M) = [df(X), df(Y)]_N$.*

Proof. Take any $g \in \mathcal{C}^\infty(N)$ and compute $df([X, Y]_M)(g)$.

$$\begin{aligned} df([X, Y]_M)_{f(p)}(g) &= [X, Y]_p(g \circ f) = \\ &= X_p(Y(g \circ f)) - Y_p(X(g \circ f)) = \\ &= X_p(df(Y(g) \circ f)) - Y_p(df(X(g) \circ f)) = \\ &= (df(X)_{f(p)})(df(Y))(g) - (df(Y)_{f(p)})(df(X))(g) = \\ &= [df(X), df(Y)]_N(g) \end{aligned}$$

□

2.1.3 The exponential map

Theorem 2.1.28. *Let M be a manifold and $X \in \mathfrak{X}(M)$. For every $p \in M$, there exists I_p, a neighbourhood of 0 in $\mathbb{R}$, such that there is a solution $\gamma : I_p \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(t) = X(\gamma(t))$.*

Proof. We prove unicity. Assume that there exist two solutions (γ_1, I_1) and (γ_2, I_2) of

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases} \tag{2.1.1}$$

Then, for any $t \in I_1 \cap I_2$, $\gamma_1(t) = \gamma_2(t)$, so there exists a maximal extension to a given solution. □

Lemma 2.1.29. *Let G be a Lie group. Then, left-invariant vector fields are complete.*

Proof. Let us take a maximal solution $\gamma : I \to G$ such that it is solution to:

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = e \in G \end{cases} \tag{2.1.2}$$

where $X \in \mathfrak{X}(G)^{L(G)}$ (meaning that X is a left-invariant vector field). We claim that $(L_g \circ \gamma) : I \to G$ is a solution to:

$$\begin{cases} \frac{d}{dt}(L_g \circ \gamma)(t) = X(g\gamma(t)) \\ (L_g \circ \gamma)(0) = g \end{cases} \tag{2.1.3}$$

Indeed, $\frac{d}{dt}(L_g \circ \gamma)(t) = d(L_g)_{\gamma(t)}(\gamma'(t)) = d(L_g)_{\gamma(t)} X(\gamma(t)) = X(L_g \circ \gamma(t)) = X(g\gamma(t))$.

Assume $I = (-a, b)$ and consider $g = \gamma(b-\varepsilon)$. Then, we have just proved that γ and $\widetilde{\gamma}$ are two integral curves of X and $\widetilde{\gamma}$ extends γ, which is a contradiction with the fact that the interval $I = (-a, b)$ was maximal. Then, I is unbounded. □

Definition 2.1.30. A *1-parameter subgroup* of a Lie group G is a smooth $\mathbb{R}$-homomorphism $\phi : (\mathbb{R}, +) \to (G, \cdot)$.

Proposition 2.1.31. *There is a one-to-one correspondence between left-invariant vector fields and* 1*-parameter subgroups.*

Proof. Let ϕ be a 1-parameter subgroup. Take $X_e = d\phi_0(\frac{\partial}{\partial t})$ and left-multiply.

On the other hand, let X be a left-invariant vector field and take the integral curve $\gamma : (\mathbb{R}, +) \to (G, \cdot)$ such that $\gamma(0) = e \in G$. It is complete by Lemma 2.1.29. It just lasts to prove that $\gamma(t + s) = \gamma(t)\gamma(s)$.

Let us denote $\gamma_1(t) = \gamma(s) \cdot \gamma(t)$ and $\gamma_2(t) = \gamma(s+t)$. It is easy to check that both $\gamma_1(t)$ and $\gamma_2(t)$ are integral curves of X:

$$\begin{aligned}
\frac{d}{dt}\gamma_1(t) &= \frac{d}{dt}(L_{\gamma(s)} \circ \gamma(t)) = \\
&= d(L_{\gamma(s)})_{\gamma(t)}(\dot{\gamma}(t)) = \\
&= d(L_{\gamma(s)})_{\gamma(t)}(X(\gamma(t))) = \\
&= X(\gamma(s)\gamma(t)) = \\
&= X(\gamma_1(t))
\end{aligned}$$

and similarly for $\gamma_2(t)$. In both cases, the initial condition is the same, because $\gamma_1(0) = \gamma(s) = \gamma_2(0)$, so we end the proof. $\square$

Definition 2.1.32. Let G be a Lie group and $\mathfrak{g}$ be a Lie algebra. The *exponential map* is defined by:

$$\begin{array}{rccc}
\exp: & \mathfrak{g} & \longrightarrow & G \\
& X & \longmapsto & \gamma^{\widetilde{X}}(1)
\end{array}$$

where where $\gamma^{\widetilde{X}} : \mathbb{R} \to G$ is the unique one-parameter subgroup of G whose tangent vector at the identity is equal to X.

Proposition 2.1.33. *Suppose* exp *is the exponential map of the Lie group G. Then:*

1. exp *is smooth.*
2. $\exp(tX) = \gamma^{\widetilde{X}})(t)$.
3. $\exp((s+t)X) = \exp(sX) \cdot \exp(sT)$.
4. $d(\exp)_0 : T_0\mathfrak{g} = \mathfrak{g} \to T_0\mathfrak{g} = \mathfrak{g}$ *is the identity.*
5. exp *is a local diffeomorphism.*
6. *If $f : G \to H$ is a Lie group homomorphism, and $F := df_e$, the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "F"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\
G \arrow[r, "f"] & H
\end{tikzcd} \tag{2.1.4}$$

Proof. 1. The existence theorem of ODE's implies smoothness with respect to initial condition, but not necessarily with respect to X. To solve this, define a vector fiels $Y \in \mathfrak{X}(\mathfrak{g} \times G)$ as

$$Y_{(x,g)} = 0_x + \widetilde{X}_g.$$

Now, the flow of Y is

$$\psi_t^Y(z,h) = (z, R_{\gamma_e^{\widetilde{X}}(t)}(h)).$$

Since $Y_{(x,g)}$ is smooth, $R_{\gamma_e^{\widetilde{X}}(t)}(h))$ is smooth too and so is $\psi_t^Y(z,h)$. Hence, as $h \cdot \gamma_e^{\widetilde{X}}(t) = R_{\gamma_e^{\widetilde{X}}(t)}(h))$ is smooth. If we evaluate the expression at $h = e, t = 1$ we obtain that $\gamma_e^{\widetilde{X}}(1) = \exp(X)$ is smooth.

2. Consider $\eta(s) := \gamma^X(s \cdot t)$. Then, $\eta'(s) = t \cdot X(\gamma^X(s \cdot t)) = X(\eta(s))$. Then, η is an integral curve of the field tX. Thus, by definition of exp, we see that $\exp(tX) = \eta(1) = \gamma^X(t)$.

3. $\exp((s+t)X) = \gamma^X(s+t) = \gamma^X(s) \cdot \gamma^X(t) = \exp(sX) \cdot \exp(tX)$.

4. In order to check it, let us compute $d\exp_0(X)$. We take $\sigma : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ such that $\sigma(0) = 0$ and $\sigma'(0) = X$. For instance, we take $\sigma(t) = tX$. We compute:

$$d\exp_0(X) = \frac{d}{dt}(\exp(tX))|_{t=0} = \frac{d}{dt}\gamma^X(t)|_{t=0} = X$$

5. Apply the Inverse Function Theorem to check that exp is a local diffeomorphism.

6. Let $X \in \mathfrak{g}$. Define $\sigma(t) = f(\exp^G(tX))$. We need to prove that $\sigma(t)$ is the integral curve of the left-invariant vector field $\widetilde{F(X)} \in \mathfrak{X}^L(H)$. By definition, $\exp^H(tF(X)) = \gamma^{\widetilde{F(X)}}(t)$. It is clear that $\sigma(0) = e_H$ and, on the other hand, $\frac{d}{dt}\sigma(t)|_{t=0} = df_{e_G}X = FX$. This proves that σ is a 1-parameter subgroup, so $f(\exp^G(X)) = \exp^H(F(X))$.

□

Example 2.1.34. Take $G = (\mathbb{R}, +)$ and $\mathfrak{g} = (\mathbb{R}^n, [\cdot,\cdot] = 0)$. Then, for any $X \in \mathfrak{g} = \mathbb{R}^n$, $\widetilde{X}_g = X$. And we have that $\exp(X) = X$.

Example 2.1.35. Take $G = (\mathrm{GL}(n,\mathbb{R}), \cdot)$ and $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$. Then, for any $X \in \mathfrak{g}$, $\widetilde{X}_g = g \cdot X$. And we have that $\exp(X) = \gamma^{\widetilde{X}}(1)$. Since $\gamma^{\widetilde{X}}(t)$ satisfies $\frac{d\gamma^{\widetilde{X}}(t)}{dt} = \widetilde{X}(\gamma^{\widetilde{X}}(t)) = \gamma^{\widetilde{X}}(t) \cdot X$, we have that $\gamma^{\widetilde{X}}(t) = e^{tX}$. Then, $\exp(X) = e^X = \sum \frac{X^k}{k!}$.

Definition 2.1.36. Let G be a matrix Lie group. Then, the *associated Lie algebra* is defined to be

$$\mathfrak{g} = \{X \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid e^{tX} \in G \ \forall t \in \mathbb{R}\}.$$

Baker–Campbell–Hausdorff formula

Consider a Lie group G and its Lie algebra $\mathfrak{g}$. Take a small neighbourhood V' around $0 \in \mathfrak{g}$ such that $\exp|_{V'} : V' \subset \mathfrak{g} \to U' \subset G$ is a diffeomorphism. Take a smaller neighbourhood U such that $U \cdot U \in U'$ and $U^{-1} \in U'$ and denote $V = \exp^{-1}(U)$ (then, $\exp^{-1}|_V : V \to U$ is still a diffeomorphism). In this construction, it is clear that, for every $X, Y \in V, \exp(X) \cdot \exp(Y) = \exp(Z)$ for some $Z \in V$.

Theorem 2.1.37. *There exists $Z = \mu(X, Y)$ such that $exp(X) \cdot \exp(Y) = \exp(Z)$ holds. $\mu : V \times V \to V$ can be written only in therms of Lie algebra operations, i.e., $X, Y, [X, Y], [X, [X, Y]], \ldots$*

Let us compute μ for $\mathbf{GL}(n, \mathbb{C})$. We already know that for $A \in \mathfrak{gl}(n, \mathbb{C}), \exp(A) = e^A = \sum \frac{A^k}{k!}$. We use its expansion at 0 to obtain:

$$e^{tA} \cdot e^{tB} = (I + tA + \frac{1}{2}t^2A^2 + \cdots)((I + sB + \frac{1}{2}s^2B^2 + \cdots) = \tag{2.1.5}$$

$$= I + tA + sB + \frac{1}{2}t^2A^2 + \frac{1}{2}s^2B^2 + tsAB + \cdots \tag{2.1.6}$$

When t and s are small, this equals

$$e^{\mu(tA,sB)} = I + \mu(tA, sB) + \frac{1}{2}\mu^2(tA, sB) + \cdots$$

Comparing both expressions, we see that up to first order, $\mu(tA, sB) = tA + sB$. But with this definition of μ, the second order is not correct, because:

$$I + \mu(tA, sB) + \frac{1}{2}\mu^2(tA, sB) = I + tA + sB + \frac{1}{2}t^2A^2 + \frac{1}{2}ts(AB + BA) + \frac{1}{2}s^2B^2$$

Then, we modify the expression of $\mu(tA, sB)$ to

$$\mu(tA, sB) = tA + sB + \frac{1}{2}tsAB - \frac{1}{2}tsBA = tA + sB + \frac{1}{2}ts[A, B]$$

And, following the same procedure, we can obtain an expression for $\mu(tA, sB)$.

Proposition 2.1.38. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then,* $\exp(tX)\cdot\exp(tY) = \exp(t(X+Y) + \frac{t^2}{2}[X,Y] + o(t^3))$ *for t sufficiently small.*[1]

To prove the proposition we need the following lemma.

Lemma 2.1.39. *Let G be a Lie group and $f : G \to \mathbb{R}$ a differentiable map. Then:*

$$\frac{d}{dt} f(g\exp(tX)) = \widetilde{X} f(g\exp(tX))$$

Proof. $\frac{d}{dt} f(g\exp(tX)) = \frac{d}{dt} f(R_{\exp(tX)}(g)) = \frac{d}{dt} f(\psi_t^{\widetilde{X}}(g)) = \widetilde{X} f(g\exp(tX))$ □

Proof of the Proposition. Apply the lemma at $g = e$ and $t = 0$ to obtain

$$\frac{d}{dt} f(\exp(tX))|_{t=0} = \widetilde{X} f(e)$$

and

$$\frac{d^2}{dt^2} f(\exp(tX))|_{t=0} = \widetilde{X}^2 f(e)$$

Then:

$$f(\exp(tX)) = f(e) + tXf(e) + \frac{1}{2}t^2X^2f(e) + o(t^3) \tag{2.1.7}$$

Now, consider

$$\begin{array}{rccc} u: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (s,t) & \longmapsto & f(\exp(tX)\cdot\exp(sY)) \end{array}$$

And do the Taylor expansion of $u(t,t)$ around $t = 0$:

$$u(t,t) = u(0,0) + t\left(\frac{\partial u}{\partial s}(s,t)|_{(0,0)} + \frac{\partial u}{\partial t}(s,t)|_{(0,0)}\right) + \tag{2.1.8}$$

$$+ \frac{1}{2}t^2\left(\frac{\partial^2 u}{\partial s^2}(s,t)|_{(0,0)} + 2\frac{\partial^2 u}{\partial s\partial t}(s,t)|_{(0,0)} + \frac{\partial^2 u}{\partial t^2}(s,t)|_{(0,0)}\right) + o(t^3) = \tag{2.1.9}$$

$$= f(e) + t\left(Xf(e) + Yf(e)\right) + \frac{t^2}{2}\left(X^2f(e) + Y^2f(e) + 2XYf(e)\right) \tag{2.1.10}$$

---

[1] The terms up to order 5 of the B-C-H formula are:
$\mu(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) - \frac{1}{24}[Y,[X,[X,Y]]] - \frac{1}{720}([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]]) + \frac{1}{360}([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]) + \frac{1}{120}([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]) + \cdots$.

Since exp is a local diffeomorphism for small t, $\exp(tX)\exp(tY) = \exp(\gamma(t))$, where $\gamma : (-\varepsilon, \varepsilon) \to U \subset \mathfrak{g}$ with $\gamma(0) = 0$. Let us chose a general $\gamma(t)$ satisfying it:

$$\gamma(t) = At + \frac{t^2}{2}B + o(t^3) \qquad A, B \in \mathfrak{g}$$

Then:

$$f(\exp(\gamma(t))) = f(\exp(tA + \frac{t^2}{2}B + o(t^3))) = \tag{2.1.11}$$

$$= f(e) + t(Xf(e) + Yf(e)) + \frac{t^2}{2}(X^2f + Y^2f + 2XYf) + o(t^3) \tag{2.1.12}$$

On the other hand:

$$f(\exp(\gamma(t))) = f(\exp(tA + \frac{t^2}{2}B + o(t^3))) = \tag{2.1.13}$$

$$= f(e) + t\widetilde{A}f(e) + \frac{t^2}{2}\widetilde{B}f(e) + \frac{t^2}{2}\widetilde{A}^2f(e) + o(t^3) \tag{2.1.14}$$

Equaling the terms at first order, we obtain

$$Xf(e) + Yf(e) = \widetilde{A}f(e) \Longrightarrow \widetilde{A} = X + Y$$

Equaling the terms at second order, we obtain

$$\frac{1}{2}\left(X^2f(e) + Y^2f(e) + 2XYf(e)\right) = \frac{1}{2}\left(\widetilde{B}f(e) + \widetilde{A}^2f(e)\right) \Longrightarrow \widetilde{B} = [X, Y]$$

□

Remark 2.1.40. If $G \cong H$, then $\mathfrak{g} \cong \mathfrak{h}$. However, $\mathfrak{g} \cong \mathfrak{h}$ does not imply that $G \cong H$. But from B-C-H we see that locally around 0 it is true.

Proposition 2.1.41. *Let G be a Lie group and $\alpha : \mathbb{R} \to G$ a continuous homomorphism. Then, α is smooth.*

Proof. We just need to prove that α is smooth around 0. Take a small neighbourhood $B_r(0) \subset \mathfrak{g}$ such that $\exp : B_{2r}(0) \to \exp(B_{2r}(0)) \subset G$ is a diffeomorphism. Consider $\beta := \exp^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to B_r(0)$, where $\varepsilon > 0$ is such that $\alpha((-\varepsilon, \varepsilon)) \subset \exp(B_r(0))$. This ε exists because α is continuous.

Now, take t such that $|t| < 2\varepsilon$. Then:

$$\exp(\beta(2t)) = \alpha(2t) = \alpha(t+t) = \alpha(t) \cdot \alpha(t) = \exp(\beta(t)) \cdot \exp(\beta(t)) = \exp(2\beta(t)),$$

implying that $\beta(2t) = 2\beta(t)$. Then, $\frac{1}{2}\beta(t) = \beta(\frac{t}{2})$ and, by induction, $\frac{1}{2^n}\beta(t) = \beta(\frac{t}{2^n})$ for any $n \in \mathbb{N}$.

Now:

$$\alpha\left(\frac{m\varepsilon}{2^n}\right) = \left(\alpha\left(\frac{\varepsilon}{2^n}\right)\right)^m = \left(\exp\left(\beta\left(\frac{\varepsilon}{2^n}\right)\right)\right)^m = \exp\left(m\beta\left(\frac{\varepsilon}{2^n}\right)\right) = \exp\left(\frac{m}{2^n}\beta\left(\varepsilon\right)\right).$$

Take the set $A = \{\frac{m}{2^n} \mid m, n \in N\}$, which is dense in $\mathbb{R}$. Since $\alpha\left(\frac{m\varepsilon}{2^n}\right) = \exp\left(\frac{m}{2^n}\beta\left(\varepsilon\right)\right)$, we can take the $t \in \mathbb{R}$ such that $|t| < 2\varepsilon$ and construct a succession $\{\frac{m\varepsilon}{2^n}\} \to t$. By continuity of α, $\alpha\left(\frac{m\varepsilon}{2^n}\right) = \exp\left(\frac{t}{\varepsilon}\beta\left(\varepsilon\right)\right)$, so α is smooth. $\square$

Proposition 2.1.42. *Let $f : G \to H$ be a continuous homomorphism between Lie groups. Then, f is smooth.*

Proof. Assume G has dimension n and take a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. Consider the map

$$\begin{array}{cccc} \phi : & \mathbb{R}^n & \longrightarrow & G \\ & (t_1, \ldots, t_n) & \longmapsto & \exp(t_1X_1)\cdots\exp(t_nX_n) \end{array}$$

whose differential at zero is:

$$\begin{array}{cccc} d\phi_0 : & \mathbb{R}^n & \longrightarrow & \mathfrak{g} \\ & (a_1, \ldots, a_n) & \longmapsto & a_1X_1 + \cdots + a_nX_n \end{array}$$

Hence, ϕ it is a diffeomorphism around 0.

Now, consider $f \circ \phi$:

$$\begin{array}{cccc} f \circ \phi : & \mathbb{R}^n & \longrightarrow & H \\ & (t_1, \ldots, t_n) & \longmapsto & f(\exp(t_1X_1)\cdots\exp(t_nX_n)) \end{array}$$

Since f is a diffeomorphism, $(f\circ\phi)(t_1, \ldots, t_n) = f(\exp(t_1X_1))\cdot\cdots\cdot f(\exp(t_nX_n))$. By Proposition 2.1.41, $f(\exp(t_iX_i))$ is smooth for any i, so $f \circ \phi$ is C^∞.

Finally, $f = (f \circ \phi) \circ \phi^{-1}$, so f is smooth. $\square$

2.1.4 Lie subgroups

A *Lie subgroup* of a Lie group G is a subgroup of G endowed with a topology and a group structure making it a Lie group and an immersed submanifold.

Definition 2.1.43.

Proposition 2.1.44. *Let G be a Lie group and H a subgroup of G that is also an embedded submanifold. Then, H is a Lie subgroup of G.*

Proof. To prove the proposition, we want to equip G with a Lie group structure. We know that the group operation $m : G \times G \longrightarrow G$ is C^∞. Then, the restriction $m|_H : H \times H \longrightarrow H \subset G$ is also smooth. Since H is embedded, $m : H \times H \longrightarrow H \subset G$ is C^∞. To conclude that H is a Lie group, we still have to check that H is closed. Take a sequence $\{h_i\}_{i\in I} \subset H$ such that $\lim_{i\to\infty} h_i = g \in G$. Since H is embedded, take a slice chart U which contains e and a smaller neighbourhood $W \subset U$ of e such that $\overline{W} \subset U$ and such that $\mu : (g_1, g_2) \mapsto g_1^{-1} \cdot g_2 : G \times G \to G$. By continuity, there exists a neighbourhood V of e such that such that $\mu : V \times V \longrightarrow W$, so if $\lim_{i\to\infty} g \cdot h_i = e$, then $\{g \cdot h_i\}_{i\in I} \subset V$ because $h_i^{-1} \cdot h_j = \underbrace{(g^{-1}h_i)^{-1}}_{\in V} \cdot \underbrace{(g^{-1}h_j)}_{\in V} \in W$.

Now, fix i and then $\lim_{j\to\infty} h_i^{-1} \cdot h_j = h_i^{-1} \cdot g \in U$ (because $\overline{W} \subset U$). Since $U \cap H$ is a slice, it is closed. Therefore, $h_i^{-1} \cdot g \in U \subset H \Longrightarrow g \in H \Longrightarrow H$ is closed. □

Theorem 2.1.45 (Closed Subgroup Theorem)**.** *Let G be a Lie group. Take $H \subset G$ a closed subgroup. Then, H is an embedded Lie group.*

Proof. We need to show that H is embedded. For this, we construct slice charts. The obvious candidate to be the Lie algebra associated to H is $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \, \forall t \in \mathbb{R}\} \subset \mathfrak{g}$.

$\mathfrak{h}$ is a vector space because:

1. $X \in \mathfrak{h} \Rightarrow tX \in \mathfrak{h} \, \forall t \in \mathbb{R}$, it is clear from the definition
2. $X, Y \in \mathfrak{h} \Rightarrow X + Y \in \mathfrak{h}$, because for any $X, Y \in \mathfrak{h}, \exp t(X+Y) = \exp(tX + tY)$. For n big, $\underbrace{\exp(\frac{t}{n}X)}_{\in H} \cdot \underbrace{\exp(\frac{t}{n}Y)}_{\in H} = \underbrace{\exp(\frac{t}{n}(X+Y) + \mathcal{O}(\frac{t^2}{n^2}))}_{\in H}$. Then, $\exp(\frac{t}{n}(X+Y) + \mathcal{O}(\frac{t^2}{n^2}))^n = \exp(t(X+Y) + \mathcal{O}(\frac{t^2}{n})) \in H$. so taking $n \longrightarrow \infty$ we obtain that $\exp(t(X+Y)) \in H$ because H is closed.

To show that H is embedded, we want to find neighbourhoods $U \subset G$ of e and $V \subset \mathfrak{h}$ of 0 such that $\exp|_V$ is a diffeomorphism and $\exp V = U \cap H$. Let us fix an inner product on $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, so we can define

$$\begin{array}{rccc} \psi : & \mathfrak{h} \oplus \mathfrak{h}^\perp & \longrightarrow & G \\ & (X, Y) & \longmapsto & \exp(X) \cdot \exp(Y) \end{array}$$

which is a diffeomorphism around 0. This gives us neighbourhoods $V \subset \mathfrak{h}$ of 0, $V' \subset \mathfrak{h}$ of 0 and $U \subset G$ of e such that $\forall g \in U \, \exists X \in V$ and $Y \in$

V' such that $g = \psi(X,Y) = \exp X \cdot \exp Y$. We want to check that $g \in U \cap H$ and that $Y = 0$.

We first see claim there exists an $\varepsilon > 0$ such that $\forall Y \in \mathfrak{h}^{\perp}$, $0 < \|Y\| \leq \varepsilon \Longrightarrow \exp Y \notin H$.

Indeed, assume that there exists a succession $\{Y_i\}_{i\in I} \subset \mathfrak{h}^{\perp}$ such that $\lim_{i\to\infty} \|Y_i\| = 0$ and $\exp(Y_i) \in H$ and we will arrive to a contradiction.

For every $i \in I$, define $Z_i := \frac{Y_i}{\|Y_i\|}$, so $\|Z_i\| = 1$. The set $\{Z_i\}_{i\in I}$ belongs to the unit sphere, a compact, so $\lim_{i\to\infty} Z_i = Z$ with $\|Z\| = 1$. Consider $t \in \mathbb{R}^+$, take $t \cdot Z_i = \frac{t}{\|Y_i\|} \cdot Y_i = (k_i + \varepsilon_i)Y_i$, where $k_i \in \mathbb{N}$ and $\varepsilon_i \in [0,1)$. Since $\lim_{i\to\infty} tZ_i = tZ$, and $\lim_{i\to\infty} \varepsilon_i Y_i = 0$ because $\lim_{i\to\infty} \|Y_i\| = 0$ and ε_i is bounded, we have that $\lim_{i\to\infty} k_i Y_i = tZ$.

By continuity of exp, we have that $\lim_{i\to\infty} \overbrace{\exp(Y_i)}^{\in H}{}^{k_i} = \lim_{i\to\infty} \exp(k_i Y_i) = \exp(tZ)$, and then $\exp(tZ) \in H$ because H is closed, and hence, $Z \in \mathfrak{h}$. But $Z \in \mathfrak{h}^{\perp}$, which implies that $Z \in \mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$, which is a contradiction with $\|Z\| = 1$.

Now, in the expression $\psi : V \times V' \subset \mathfrak{h} \oplus \mathfrak{h}^{\perp} \longrightarrow U \subset G$, take $V' \cap \{Y \in \mathfrak{h}^{\perp} \mid \|Y\| < \varepsilon$ for ε as in the claim$\}$. This means that for $g \in U \cap H$, $g = \exp X \cdot \exp Y \Longrightarrow Y = 0$, which implies that $g \in \exp(V)$. This proves that $U \cap H = \exp V$. □

Remark 2.1.46. A consequence of the Closed Subgroup Theorem is that matrix Lie groups are embedded Lie subgroups of $\mathrm{GL}(n, \mathbb{C})$.

Proposition 2.1.47. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Take H a Lie subgroup of G. Then, the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$.*

Proof. Take the inclusion map $i : H \longrightarrow G$, which is a Lie group morphism. The differential $di_e : \mathfrak{h} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism. □

Theorem 2.1.48. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then, there exists a unique connected Lie subgroup H of G such that its Lie algebra is $\mathfrak{h}$.*

Proof. The idea is to consider a distribution $D = \langle \widetilde{X_1}, \ldots, \widetilde{X_k} \rangle$, where $\{X_1, \ldots, X_k\}$ is a basis of $\mathfrak{h}$ and $\widetilde{X_i}$ are the corresponding left-invariant vector fields. Integrating the distribution will give the manifold (the Lie group). □

2.1.5 Integration of differential distributions

Lemma 2.1.49. *Let $X, Y \in \mathfrak{X}(M)$. Then, $[X,Y] = 0 \iff XY - YX = 0 \iff \phi_s^X \circ \phi_t^Y = \phi_t^Y \circ \phi_s^X$, where ϕ_r^Z is the flow of the vector field Z at time r.*

Proposition 2.1.50. *Let X, Y be two vector fields on a manifold M such that $[X,Y] = 0$. Then, there exists a smooth map $F : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \longrightarrow M$ such that:*

1. $F(0,0) = p$
2. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial s}\right) = X(F(s,t))\big|_p$
3. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial t}\right) = Y(F(s,t))\big|_p$

Proof. Take $F = \phi_s^X \circ \phi_t^Y(p)$, which satisfies $\phi_s^X \circ \phi_t^Y = \phi_t^Y \circ \phi_s^X$ by previous Lemma 2.1.49 and, then:

1. $F(0,0) = p$
2. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial s}\right) = \frac{d}{ds}\left(\phi_s^X\right)\left(\phi_s^X \circ \phi_t^Y\right)(p) = X(F(s,t))$
3. $dF_{(s,t)}\big|_p \left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}\left(\phi_t^Y\right)\left(\phi_t^Y \circ \phi_s^X\right)(p) = Y(F(s,t))$

□

Definition 2.1.51. The map $F = \phi_s^X \circ \phi_t^Y$ is called *integral surface.*

Definition 2.1.52. A *differential distribution* D of rank k is the object that satisfies the following properties:

- For every $p \in M$, $D_p \leq T_pM$, i.e. D_p is a subspace of dimension k.
- For every $p \in M$, there exists a neighbourhood $U \subset M$ of p and $X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that if $q \in U$, then $\langle X_1(q), \cdots, X_l(q)\rangle = D_q$.

Example 2.1.53. In $M = R^n$, $D = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\rangle$ with $k \leq n$ is a distribution of rank k and, in fact, is R^k at every point.

Example 2.1.54. In $M = R^3$, $D = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\rangle$ is a distribution of rank 2 and, in fact, it is a plane at every point.

Remark 2.1.55. A distribution of rank 1 is a vector field.

Definition 2.1.56. Let D be a differential distribution of rank k on a manifold M. Then, D is *integrable* if there exists a k-dimensional embedded submanifold $S \subset M$ such that for all $p \in S$, $T_pS = D_p$.

In Example 2.1.53, the distribution is integrable, while in Example 2.1.54, it is not integrable.

Definition 2.1.57. A distribution D on a manifold M is *involutive* if $\forall X, Y$ vector fields locally defined on an open neighbourhood U, and for every $p \in U$, $[X, Y]_p \in D_p$.

Theorem 2.1.58 (Frobenius Theorem)**.** *A distribution D is integrable if and only if it is involutive.*

Proof. From left to right: Let S be an integrable submanifold. Suppose X, Y are two vector fields locally defining the distribution. Then, $X, Y \in \mathfrak{X}(S)$ (locally). Therefore, $[X, Y] \in D$ (locally), so D is involutive.

From right to left: It is clear that we can integrate the distribution if the vector fields $X_1, \ldots, X_k$ locally defining D satisfy $[X_i, X_j] = 0$ for all i, j. Indeed, we can integrate them to $F : (t_1, \ldots, t_k) \longmapsto \phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_k}^{X_k} : (-\varepsilon, \varepsilon)^k \longrightarrow M$. Since $[X_i, X_j] = 0$, by Lemma 2.1.49 we know that $\phi_{t_i}^{X_i} \circ \phi_{t_j}^{X_j}$ commute. So, we want to find some $Y_1, \ldots, Y_k$ locally defining D and such that $[Y_i, Y_j] = 0$ for all i, j.

Let $p \in M$ and $U \subset M$ a neighbourhood of p. Take $X_1, \cdots, X_k$ defining D. At p, $X_i(p) = \frac{\partial}{\partial x_i}$ for all $= 1, \ldots, k$. In U, consider $B = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$. We have that $D_q \pitchfork B_q$ (they are transversal) for any $q \in U$ (maybe shrinking U) because transversality is an open condition. Now, we will prove that we can find an $Y_1, \ldots, Y_k$ such that $Y_i = \frac{\partial}{\partial x_i} + \sum_{l \geq k+1}^{n} a_{il} \frac{\partial}{\partial x_l}$ and $[Y_i, Y_j] = 0$.

Write, for all $i = 1, \ldots, k$, X_i as $X_i \sum_{l=1}^{n} c_{il} \frac{\partial}{\partial x_l}$ (we can always write it this way). The matrix (c_{il}) is invertible in U. Take $Y_i = \sum_{l=1}^{n} d_{li} \frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_i}$, where (d_{il}) is the inverse of (c_{il}). Let us check now that $[Y_i, Y_j] = 0$.

$$[Y_i, Y_j] = [\sum_{l=1}^{n} d_{li} \frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_i}, \sum_{s=1}^{n} d_{sj} \frac{\partial}{\partial x_s} + \frac{\partial}{\partial x_j}] = \tag{2.1.15}$$

$$= \underbrace{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}]}_{=0} + \underbrace{[\sum_{l=1}^{n} d_{li} \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_j}]}_{\in B} + \tag{2.1.16}$$

$$+ \underbrace{[\frac{\partial}{\partial x_i}, \sum_{s=1}^{n} d_{sj} \frac{\partial}{\partial x_s}]}_{\in B} + \underbrace{[\sum_{l=1}^{n} d_{li} \frac{\partial}{\partial x_l}, \sum_{s=1}^{n} d_{sj} \frac{\partial}{\partial x_s}]}_{\in B} \tag{2.1.17}$$

Then, $[Y_i, Y_j] \in B \cap D$, which implies that $[Y_i, Y_j] = 0$. □

Definition 2.1.59. A *leaf* of an integrable distribution is a maximal integrable submanifold.

Remark 2.1.60. In the definition of a leaf Λ of an integrable distribution, *maximal* means that, if there exists an integrable submanifold S such that $\Lambda \cap S \neq 0$, then $S \subset \Lambda$.

Remark 2.1.61. The leaf Λ_p at a point p in a manifold with an integrable distribution can be defined as

$$\Lambda_p = \{q \in M \mid \text{you can reach } q \text{ from } p \text{ following paths in the integrable submanifold}\}.$$

Proof of Theorem 2.1.48. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \leq \mathfrak{g}$. Take $D = \{\widetilde{X_1}, \ldots, \widetilde{X_k} \mid X_1, \ldots, X_k \in \mathfrak{h}\}$. Then, D is involutive because $\mathfrak{h}$ is a subalgebra. Therefore, we can integrate. Take $H = \Lambda_e$ (the leaf of the identity element) and check that H is indeed a Lie group. Let $h_1, h_2 \in H = \Lambda_e$. Then, $h_1 \cdot h_2 = L_{h_1}(h_2) \in L_{h_1}(\Lambda_e) = \Lambda_{h_1} = \Lambda_e = H$. And $h^{-1} = h_1^{-1} \cdot e \in L_{h_1^{-1}}(\Lambda_e) = \Lambda_{h_1} = \Lambda_e = H$. □

Chapter 3

Lie Group Actions

This chapter follows [Aud91] and [Bry95].

3.1 Definition of Lie Group Action

Definition 3.1.1. Let M be a smooth manifold and G a Lie group. A *Lie group action* is a smooth mapping $\alpha : G \times M \longrightarrow M$ such that:

1. The induced mapping $\alpha_g : m \longmapsto \alpha(g, m) : M \longrightarrow M$ is a diffeomorphism for every $g \in G$.

2. For every $g_1, g_2 \in G, m \in M$, $\alpha(g_1 \cdot g_2, m) = \alpha(g_1, \alpha(g_2, m))$.

Example 3.1.2. Let G be a Lie group. G acts on itself by left-translations:

$$\begin{array}{rccll} \alpha : & G \times G & \longrightarrow & G & \\ & (g_1, g_2) & \longmapsto & L_{g_1}(g_2) & = g_1 \cdot g_2 \end{array}$$

Indeed, $\alpha_g = L_g$ is a diffeomorphism, with inverse $L_{g^{-1}}$, and satisfies $L_{g \cdot h}(k) = (g \cdot h) \cdot k = g \cdot (h \cdot k) = L_g(h \cdot k)$ for any $g, h, k \in G$.

Example 3.1.3. Let G be a Lie group. Right multiplication by an element of G ($R_g : h \longmapsto h\dot{g}$) is an action.

Example 3.1.4. From the combination of the previous two examples and the fact that the inverse of $g \in G$ is in G, we have that conjugation ($L_g \circ L_{g^{-1}}$) is an action.

Example 3.1.5. Suppose M is a compact manifold and $X \in \mathfrak{X}(M)$ (or, simply, suppose X is complete and, hence, its flow is defined for all t). The flow of X

defines an $\mathbb{R}$-action:

$$\begin{array}{rccc} \alpha: & \mathbb{R}\times M & \longrightarrow & M \\ & (t,p) & \longmapsto & \varphi_t^X(p) \end{array}$$

Where φ_t^X is the flow of X, i.e., the solution of $\frac{d\varphi_t}{dt} = X(\varphi_t)$. Indeed, φ_t^X is a diffeomorphism and $\varphi_{s+t}^X(p) = \varphi_s^X \circ \varphi_t^X(p)$.

Example 3.1.6. Take $G = \mathrm{GL}(n,\mathbb{R})$ and $M = \mathbb{R}^n$. Then, $\mathrm{GL}(n,\mathbb{R})$ acts on $\mathbb{R}^n$ by the usual matrix multiplication:

$$\begin{array}{rccc} \alpha: & \mathrm{GL}(n,\mathbb{R})\times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (A,x) & \longmapsto & A\cdot x \end{array}$$

It is a group action because $\alpha(A,\alpha(B,x)) = A(Bx) = (AB)x = \alpha(AB,x)$ and $\alpha_A^{-1} = \alpha_{A^{-1}}$.

Example 3.1.7. Take $G = O(n,\mathbb{R})$ and $M = S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subset \mathbb{R}^n$. Then, $O(n,\mathbb{R})$ acts on S^{n-1} by the usual matrix multiplication.

$$\begin{array}{rccc} \alpha: & O(n,\mathbb{R})\times S^{n-1} & \longrightarrow & S^{n-1} \\ & (A,\vec{u}) & \longmapsto & A\cdot \vec{u} \end{array}$$

It is a Lie group action because, as in the previous example, matrix multiplication satisfies the action conditions.

Exercise 3.1.8. Take $G = S^1$ and $M = S^2$. Prove that S^1 acts on S^2 (by rotations).

Exercise 3.1.9. Take $G = S^1 = \{t \in \mathbb{C} \mid \|t\| = 1\}$ and $M = S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \mid \|z_1\|^2 + \|z_2\|^2 = 1\} \subset \mathbb{R}^4$. Prove that $\alpha : (t,(z_1,z_2)) \longmapsto (t^{m_1}z_1, t^{m_2}z_2)$ defines an action for any $m_1, m_2 \in \mathbb{Z}$.

Exercise 3.1.10. Take $G = S^1 \cong SO(2,\mathbb{R})$ and $M = \mathbb{RP}^1$. Prove that the following mapping is an action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

The affine form of this map, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \frac{ax+by}{cx+dy}$, is known as the Möbius transformation.

Definition 3.1.11. Let $\alpha : G \times M \to M$ be a Lie group action. For each $m \in M$, the orbit of m is the set

$$\mathcal{O}(m) = G\cdot m = \{g\cdot m \mid g \in G\} \subset M$$

Definition 3.1.12. An action $\alpha : G \times M \to M$ is *transitive* if there is only one orbit in the manifold. In other words, if $\forall x, y \in M$, $\exists g \in G$ such that $y = \alpha(g, x) = g \cdot x$.

Definition 3.1.13. Let $\rho : G \times M \to M$ be a Lie group action. For each $m \in M$, the *stabilizer* or *isotropy group* at m is the set of elements of G that fix m, i.e.:

$$G_m = \{g \in G \mid g \cdot m = m\} \subset G$$

Lemma 3.1.14. *For every $m \in M$, the stabilizer G_m is a subgroup of G (in fact, a closed subgroup).*

Proof. Let us see first that G_m is a subgroup of G. Take $m \in M$.

1. For $g_1, g_1 \in G_m$, $(g_1 \cdot g_2) \cdot m = g_1 \cdot (g_2 \cdot m) = g_1 \cdot m = m \Longrightarrow g_1 \cdot g_2 \in G_m$.
2. $e \cdot m = m \Longrightarrow e \in G_m$.
3. For $g \in G_m$, $g^{-1} \cdot m = g^{-1} \cdot (g \cdot m) = (g^{-1} \cdot g) \cdot m) = m \Longrightarrow g^{-1} \in G_m$.

Consider now the continuous map $\alpha_m : G \longrightarrow M$ defined by $\alpha_m(g) := \alpha(g, m)$. Then, $\alpha_m{-1}(g) = G_m \Longrightarrow G_p$ is closed. ☐

Proposition 3.1.15. *The isotropy groups of points on the same orbit are conjugated. In other words, if $x, y \in M$, $g \in G$ and $y = g \cdot x$, then $G_y = gG_xg^{-1}$.*

Definition 3.1.16. If $G_m = \{e\} \forall m \in M$, i.e., all the isotropy groups are trivial, the action is called *free*. It is called *locally free* if all the G_m's are discrete.

Definition 3.1.17. An action is *effective* if $\bigcap_{m \in M} G_m = \{e\}$.

The manifold M can be partitioned in orbits because, for every $m \in M$, the evaluation map $ev_m : g \mapsto g \cdot m$ induces a bijection between the quotient G/G_m and the orbit $G \cdot m$. The quotient set M/G is precisely the set of orbits in which M decomposes. We state some topological results that will be necessary to prove that the quotient is a manifold.

Lemma 3.1.18. *Suppose M is a locally compact Hausdorff space, equipped with a continuous right action of a topological group H. Then the following statements are equivalent:*

- *The orbit space ${}^{M}\!/_{H}$ is Hausdorff.*
- *For any compact subset $C \subset M$, the set CH is closed.*

Proof. Suppose that the orbit space ${}^{M}\!/_{H}$ is Hausdorff. Take $C \subset M$ compact. Then, the class of C in ${}^{M}\!/_{H}$, i.e. $\pi(C)$ is compact. Since ${}^{M}\!/_{H}$ is Hausdorff $\pi(C)$ is closed. As $CH = \pi^{-1}\pi(C)$, CH is closed.

Now, suppose that for any compact subset $C \subset M$, the set CH is closed. Since $\{m\}$ is compact, the orbit mH is closed. Assume $x_1, x_2 \in X = {}^{M}\!/_{H}$ are distinct points. Choose $m_j \in \pi^{-1}(x_j)$. Then, $x_j = m_jH$, with $m_1H \cap m_2H = \emptyset$. The complementary set V of m_2H in M is open, right H-invariant and contains m_1H. Choose an open neighborhood U_1 of m_1 in M such that $\overline{U}_1$ is compact and contained in V. Then, the set $\overline{U}_1H$ is closed and still contained in V. Its complementary set V_2 is open and contains m_2H. Then, $\pi(V_2)$ is open in X and contains x_2.

On the other hand, $V_1 = U_1H$ is the union of the open sets U_1h, so it is open in M. V_1 contains m_1. so that $\pi(V_1)$ is an open neighborhood of x_1 in X. The sets V_1 and V_2 are right H-invariant and disjoint. It follows that the sets V_1 and V_2 are disjoint open subsets of X containing the points x_1 and x_2, respectively, implying that the orbit space ${}^{M}\!/_{H}$ is Hausdorff. □

Definition 3.1.19. Assume that M is a smooth manifold and that a Lie group H is right-acting on the manifold $M \times H$, with an action given by $(x, g) \cdot h = (x, gh)$. We will say that such an action is of *trivial principal fiber bundle* type. We call that it is of *principal fiber bundle (PFB)* if the following two conditions are fulfilled:

- Every point m of M has an open H-invariant neighborhood U such that the right H-action on U is of trivial PFB type.
- Whenever C is a compact subset of M, then CH is closed.

Remark 3.1.20. Lemma 3.1.18 shows that the second condition in the definition is equivalent to saying that the quotient space ${}^{M}\!/_{H}$ is Hausdorff.

We want now to prove a theorem that shows that, under certain conditions (certain smooth actions), the quotient space admits a unique natural structure of smooth manifold. It will be necessary to prove 3.1.24 afterwards.

Theorem 3.1.21. *Let H be a right action on M be of Principal Fiber Bundle type. Then ${}^{M}\!/_{H}$ carries a unique structure of $\mathcal{C}^\infty$-manifold (compatible with the topology) such that the canonical projection $\pi : M \longrightarrow {}^{M}\!/_{H}$ is a smooth submersion. If $m \in M$, the tangent map $T_m\pi : T_mM \longrightarrow T_{\pi(m)}({}^{M}\!/_{H})$ has kernel $T_m(mH)$, the tangent space of the orbit mH at m. Accordingly,* *it induces a linear isomorphism from* $T_mM/_{T_m(mH)}$ *onto* $T_{\pi(m)}(M/_H)$. *Finally,* $\pi^*: f \longmapsto f\circ\pi$ *restricts to a bijective linear map from* $\mathcal{C}^\infty(M/_H)$ *onto* $\mathcal{C}^\infty(M)^H$.

Remark 3.1.22. The dimension of $M/_H$ equals $\dim M - \dim H$.

Proof. For the proof, see Section 12 in `https://www.staff.science.uu.nl/~ban00101/lie2012/lie2010.pdf`. □

Definition 3.1.23. For each $x \in M$, the *orbit map* is defined as

$$\begin{array}{cccc} f_x: & G & \longrightarrow & M \\ & g & \longmapsto & g\cdot x \end{array}$$

and is a smooth map. Its differential at $e \in G$ is a map $d(f_x)_e : T_eG = \mathfrak{g} \to T_xM$

Theorem 3.1.24. *The orbit map* $f_x : G \longmapsto M$ *considered at the quotient space* G/G_x, *i.e.,* $\bar{f}_x : G/G_x \longmapsto M$ *is an injective immersion.*

Proof. Since G_x is a closed subgroup, by Theorem 2.1.45, it is a Lie subgroup, so G/G_x it is a smooth manifold. Now:

1. First, we see that $\bar{f}_x$ is injective: $[g_1] = [g_2]$ for any $g_1, g_2 \in G$ if and only if $g_1 = a\cdot g_2$ for some $a \in G_x$. Let $g_1, g_2 \in G/G_x$ such that $g_1\cdot x = g_2\cdot x$. Then, $x = g_1^{-1}\cdot g_2\cdot x$, so $g_1^{-1}\cdot g_2 \in G_x$ and, then, $[g_1] = [g_2]$. Therefore, $\bar{f}_x$ is injective.

2. Then, we see that $\bar{f}_x$ is an immersion. We simply have to check that the differential $d(\bar{f}_x)_g : T_g(G) \longrightarrow T_{g\cdot x}(G\cdot x)$ is injective and we can do it evaluating its kernel. By invariance (because we can always postcompose with $d(l_{g^{-1}})$, which is an isomorphism), it is only necessary to study the case $g = e$. Then, we look at the kernel of $d(\bar{f}_x)_e$. This kernel is $\{X \in \mathfrak{g} \mid X_x = 0\}$ and it is exactly the Lie algebra $\mathfrak{g}_x$ of G_x.

□

Corollary 3.1.25. *Let* G *be a compact Lie group. Then, the orbits of any* G*-action are smooth submanifolds.*

Proof. A proper injective immersion is an embedding and we already proved that $\bar{f}_x : G/G_x \longmapsto M$ is an injective immersion. By compactness of G, we have that the map is proper, and $G/G_x \cong G\cdot x$. □

Example 3.1.26. Take $G = S^1$ and $M = S^2$ and consider the action α in Exercise 3.1.8. The diffeomorphism α_θ can be interpreted as the planar rotation of S^2 of angle θ around the z-axis (in cylindrical coordinates (r, θ, z)). Then, it is clear that M/G, the set of orbits, is the interval $[-1, 1]$.

Example 3.1.27. Like in Example 3.1.5, the action of $\mathbb{R}$ on M given by the flows of X induces a natural partition of M into orbits of the form $\gamma_x = \{(\phi_t^X(x)) \mid t \in \mathbb{R}\}$. This is the motivation of the definition of orbit in Dynamical Systems.

Chapter 4

Introduction to Symplectic and Poisson Geometry

This chapter follows [Aud91], [Can01], [DZ05] and [LPV13]. The examples in the Poisson Geometry Section are obtained from [Mir14].

4.1 Symplectic Geometry

We start this section by revising the Lie algebra of symplectic manifolds, which is that of tangent spaces.

4.1.1 Symplectic Linear Algebra

Definition 4.1.1. Let V be a vector space. Let $\omega : U \times U \longrightarrow \mathbb{R}$ be a skew-symmetric bilinear 2-form, i.e. $\omega \in \bigwedge^2 U^*$. The form ω is *non-degenerate* if

$$\omega(v, u) = 0 \ \ \forall u \in U \Longrightarrow v = 0.$$

Example 4.1.2. Take $U = \mathbb{R}^2$ with the standard coordinates (x, y). The form $\omega = dx \wedge dy$ is non-degenerate.

Example 4.1.3. Take $V = F + F^*$. The form ω that acts by:

$$\omega((x, \varphi), (x', \varphi)) = \varphi(x') - \varphi'(x)$$

is skew-symmetric and non-degenerate.
If we consider the basis $e_{ij} = (e_i, e_j^*)$ for V, with $e_i \in F$ and $e_j^* \in F^*$, we can

compute:

$$\omega(e_{ij}, e_{kl}) = e_j^*(e_k) - e_l^*(e_i) = \tag{4.1.1}$$

$$= \begin{cases} 1 \text{ if } j = k \text{ and } l \neq i \\ 0 \text{ if } j = k \text{ and } l = i \\ -1 \text{ if } j \neq k \text{ and } l = i \\ 0 \text{ if } j \neq k \text{ and } l \neq i \end{cases} \tag{4.1.2}$$

Then, $\omega(e_{ij}, e_{jk}) = e_j^*(e_j) = 1 \neq 0$, implying that the form ω is non-degenerate.

Example 4.1.4. Consider $\mathbb{C}^n$ with the standard Hermitian form ω that acts the following way. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and consider elements (X, Y) in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then, $\omega(X, Y) = \Im\langle X, Y\rangle = \Im X^t \bar{Y}$. This form is:

- Skew-symmetric, because $\Im\langle X, Y\rangle = -\Im\overline{\langle X, Y\rangle} = -\Im\langle Y, X\rangle$
- Non-degenerate, because if $\Im\langle X, Y\rangle = 0 \ \forall Y$, then $\Im\langle X, iY\rangle = 0 \ \forall i \Longrightarrow$ $\Longrightarrow \langle X, Y\rangle = 0 \forall Y \ \ h = 0$.

Remark 4.1.5. In practice, the condition of non-degeneracy can be translated into the following analytical condition. Take the matrix of the 2-form ω in some coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in such a way that $\omega = \sum \omega_{ij} \cdot dx_i \wedge dy_j$. Then, ω is non-degenerate if and only if $\det(\omega_{ij}) \neq 0$.

Remark 4.1.6. The condition of non-degeneracy of ω is also equivalent to condition for the map

$$\begin{array}{rccc} \#: & TM & \longrightarrow & T^*M \\ & X & \longmapsto & \iota_X \omega \end{array}$$

of being an isomorphism.

Remark 4.1.7. Another equivalent condition of the non-degeneracy of ω is the following:

$$\omega \in \Omega^2(M^{2n}) \text{ is non-degenerate } \Longleftrightarrow \ \omega^n = \omega \wedge \cdots \wedge \omega \text{ is a volume form.}$$

Proposition 4.1.8. *Let M^{2n} be a compact manifold without boundary. A 2-form ω on M can not be exact and non-degenerate at the same time.*

Proof. Let us assume that ω is exact and we will arrive to contradiction. Suppose $\omega = d\beta$ for some $\beta \in \Omega^1(M)$. Then:

$$d(\beta \wedge \omega^{n-1}) = d\beta \wedge \omega^{n-1} + (-1)^1 \beta \wedge d(\omega^{n-1}) = \omega \wedge \omega^{n-1} - \beta \wedge d(d\beta)^{n-1} = \omega^n.$$

Now, since ω is non-degenerate, it is a volume form and, then $0 < \int_M \omega^n$. But, on the other hand

$$\int_M \omega^n = \int_M d(\beta \wedge \omega^{n-1}) = \int_{\partial M} \beta \wedge \omega^{n-1} = 0$$

where we applied Stokes Theorem and that $\partial M = \emptyset$. □

4.1.2 Symplectic Manifolds

Definition 4.1.9. Given an even dimensional manifold M^{2n}, we say a smooth 2-form ω is a *symplectic form* if ω is closed ($d\omega = 0$) and non-degenerate ($\forall \alpha \in \Omega^1(M), \exists! X \in \mathfrak{X}(M)$ that solves $\iota_X \omega = \alpha$).

Example 4.1.10. If λ is the Liouville 1-form on T^*M, then $\omega = d\lambda$ is symplectic.

The non-degenerate forms in Examples 4.1.2, 4.1.3 and 4.1.4 are symplectic forms.

Remark 4.1.11. By Remark 4.1.7, if ω is a symplectic form on a manifold M, then it is a volume form and, hence, M is orientable. Nevertheless, not all orientable manifolds admit a symplectic form. For instance, S^4 is an orientable manifold that does not admit any symplectic form. Assume ω is symplectic, so that $d\omega = 0$ and $\omega \in \Omega^2(S^4) \implies [\omega] \in \mathcal{H}^2(S^4), [\omega] \neq [0]$. But $\mathcal{H}^2(S^4) = 0 \implies [\omega] = [0]$, so we arrive to contradiction.

Definition 4.1.12. A *symplectic manifold* is a pair (M, ω) such that M is a differential manifold and ω is a closed non-degenerate 2-form on M.

Example 4.1.13. Any orientable surface is a symplectic manifold.

Proposition 4.1.14. *If M^{2n} is a compact smooth manifold and admits a symplectic structure, then $\mathcal{H}^2(M) \neq 0$.*

Proof. Recall that $\mathcal{H}^2_{DR}(M) = {\ker(d : \Omega^2(M) \to \Omega^3(M))} \big/ {\mathrm{Im}(d : \Omega^1(M) \to \Omega^2(M))}$ Suppose that M is compact and admits a symplectic structure ω, which satisfies $d\omega = 0$. But, as we proved in Proposition 4.1.8, a 2-form can not be exact if it is non-degenerate at the same time, so the class of ω within the quotient $\mathcal{H}^2_{DR}(M)$, i.e., $[\omega] = 0$ can not be equal to 0. □

Corollary 4.1.15. *S^{2n} does not admit a symplectic structure if $n > 1$.*

Theorem 4.1.16. *Let (M^{2n}, ω) be a symplectic manifold. Then, for all $p \in M$, there exists a local coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ such that $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ in a neighbourhood U of p.*

Proof. We use the exponential map. Let p be a point in M and

$$\begin{array}{rccc} \exp_p : & U \subset T_pM & \longrightarrow & V \subset M \\ & u & \longmapsto & \gamma_u(1) \end{array}$$

is a local diffeomorphism. Locally, (M, ω) is symplectic, so T_pM is symplectic with symplectic form $\widetilde{\omega}_1 = \exp^* \omega$. We express $\widetilde{\omega}_1$ in a symplectic base $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ and we get $\widetilde{\omega}_0$, which satisfies:

- $\widetilde{\omega}_0(e_i, f_j) = \delta_{ij}$
- $\widetilde{\omega}_0(e_i, e_j) = 0$
- $\widetilde{\omega}_0(f_i, f_j) = 0$

In this symplectic base, if J is the matrix $\begin{pmatrix} 0 & -In \\ In & 0 \end{pmatrix}$, then $\widetilde{\omega}_0(u, v) = u^T \cdot J \cdot v$. Now, we define $\omega_0 := (\exp^{-1})^* \widetilde{\omega}_0$ and we apply the *Moser's trick*. We define the path

$$\omega_t = (1-t)\omega_0 + t\omega_1 \tag{4.1.3}$$

which is a path of closed forms, since $d\omega_t = (1-t)d\omega_0 + td\omega_1 = 0$ because $d\omega_0 = d\omega_1 = 0$. The forms ω_t are locally non-degenerated, because $\det(\omega_{ij}(p)) \neq 0$ and, since det is a continuous map, there exists a neighbourhood U of p such that $\det(\omega_{ij}(q)) \neq 0$ for any $q \in U$.

By Poincaré Lemma, $\omega_0 - \omega_1 = d\beta$ (because ω_1 and ω_0 are closed and d is linear). So $\omega_0 - \omega_1$ is closed and, then, locally exact. Then, there exists a vector field X_t such that $\iota_{X_t}\omega_t = -\beta$. Let φ_t be the flow of X_t and let us prove that $(\varphi_t)^*\omega_t = \omega_0 = (\varphi_1)^*\omega_1$:

$$\frac{d}{dt}\left(\varphi_t^*\omega_t\right) = \varphi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \tag{4.1.4}$$

$$= \varphi_t^*\left(-\omega_0 + \omega_1 + d\iota_{X_t}\omega_t + \iota_{X_t}d\omega_t\right) = \tag{4.1.5}$$

$$= \varphi_t^*\left(d\beta + d(-\beta) + 0\right) = \tag{4.1.6}$$

$$= \varphi_t^*(0) = 0 \tag{4.1.7}$$

Then, $\varphi_t^*\omega_t$ is constant and, since $\varphi_0^*\omega_0 = 0$, we have that $(\varphi_t)^*\omega_t = \omega_0$. □

Theorem 4.1.17. *Let S be an orientable compact surface and ω_0, ω_1 two symplectic forms on S. Then, the pair (S, ω_0) is equivalent to the pair (S, ω_1) (i.e., it exists a diffeomorphism $\varphi : S \to S$ s.t. $\varphi^*\omega_1 = \omega_0$) if and only if $[\omega_0] = [\omega_1]$, where $[\,]$ denotes the equivalence class in $\mathcal{H}^2_{DR}(S)$.*

Proof. Let ω_0, ω_1 be two symplectic forms such that $[\omega_0] = [\omega_1]$. Then, $\omega_1 - \omega_0 = d\beta$ for some $\beta \in \Omega^1(S)$. Now, consider the path of forms $\omega_t = (1-t)\omega_0 + t\omega_1$, which satisfies $d\omega_t = 0$. Since S is of dimension 2 and ω_0, ω_1 are 2-forms, $\omega_1 = f \cdot \omega_0$ for some smooth function f. Then, $\omega_t = (1-t)\omega_0 + tf\omega_0 = (1-t(1-f))\omega_0$. If $f > 0$, by convexity $\omega_t > 0$. Let φ_t be the flow of X_t and follow as in the proof of Theorem 4.1.16 considering now that φ_t is defined for every t by compactness of S. $\square$

Definition 4.1.18. Let $H \in \mathcal{C}^\infty$ be a smooth function on a symplectic manifold (M, ω) (in Physics, a Hamiltonian, the function of total energy). The Hamiltonian vector field X_H is defined as the only solution of $\iota_{X_H}\omega = -dH$.

Example 4.1.19. Take $(\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$. Let us write the flow of X_H:

$$X_H = \sum_{i=1}^{n} X_H^{x_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} X_H^{y_i} \frac{\partial}{\partial y_i} \tag{4.1.8}$$

So, on the one hand:

$$\iota_{X_H}\omega = \omega(X_H, \cdot) = \sum_{i=1}^{n} X_H^{x_i} dy_i + \sum_{i=1}^{n} -X_H^{y_i} dx_i \tag{4.1.9}$$

And, on the other hand:

$$-dH = \sum_{i=1}^{n} -\frac{\partial H}{\partial x_i} dx_i + \sum_{i=1}^{n} -\frac{\partial H}{\partial y_i} dy_i \tag{4.1.10}$$

This leads to the Hamiltonian equations:

$$\begin{cases} X_H^{x_i} dy_i = \dot{x}_i = -\frac{\partial H}{\partial y_i} dy_i \\ X_H^{y_i} dx_i = \dot{y}_i = \frac{\partial H}{\partial x_i} dx_i \end{cases} \tag{4.1.11}$$

Definition 4.1.20. A diffeomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ between symplectic manifolds is called a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$.

Lemma 4.1.21. *Symplectomorphisms are volume preserving.*

Proof. Let $\varphi : (M_1^{2n}, \omega_1) \to (M_2^{2n}, \omega_2)$ be a symplectomorphism between symplectic manifolds. Then, $\omega_1^n \in \Omega 2n(M_1), \omega_2^n \in \Omega 2n(M_2)$ are volume forms, i.e., they are not zero. Hence:

$$\mathrm{Vol}(M_1) = \int_{M_1} \omega_1^n = \int_{M_1} (\varphi^*\omega_2)^n = \int_{\varphi^{-1}(M_2)} \varphi^*(\omega_2^n) = \int_{M_2} \omega_2^n = \mathrm{Vol}(M_2) \tag{4.1.12}$$

$\square$

Remark 4.1.22. Let $(\Sigma_1^2, \omega_1), (\Sigma_2^2, \omega_2)$ be two symplectic compact surfaces, equipped with area forms. Then, there exists a symplectomorphism $\varphi : \Sigma_1 \to \Sigma_2$ if and only if:

1. $\Sigma_1 \cong \Sigma_2$ ($\iff \mathcal{X}(\Sigma_1) = \mathcal{X}(\Sigma_2)$).
2. $\int_{\Sigma_1} \omega_1 = \int_{\Sigma_2} \omega_2$.

Definition 4.1.23. The *Poisson Bracket* associated to a symplectic manifold (M, ω) is defined by

$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & (f, g) & \longmapsto & \omega(X_f, X_g) \end{array}$$

Proposition 4.1.24. *The Poisson Bracket satisfies:*

1. $\{f, g\} = -\{g, f\}$ *(anti-symmetry)*
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ *(Jacobi identity)*
3. $\{f, gh\} = h\{f, g\} + g\{f, h\}$ *(Leibniz rule)*

Proof. 1. $\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$, by anti-symmetry of ω

2. Recall that if $\omega \in \Omega^k(M)$, then $d\omega \in \Omega^{k+1}(M)$ and:

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_k) + \tag{4.1.13}$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_k) \tag{4.1.14}$$

Hence:

$$\begin{aligned} & d\omega(X_f, X_g, X_h) = && (4.1.15)\\ &= X_f \omega(X_g, X_h) - X_g \omega(X_f, X_h) + X_h \omega(X_f, X_g) - && (4.1.16)\\ &\quad - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f) = && (4.1.17)\\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\} - && (4.1.18)\\ &\quad - \{\{f, g\}, h\} + \{\{f, h\}, g\} - \{\{g, h\}, f\} = && (4.1.19)\\ &= 2\left(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\right) && (4.1.20) \end{aligned}$$

Since $d\omega = 0$, $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

3. $X_f(g) = dg(X_f) = -\iota_{X_g}\omega(X_g) = \omega(X_f, X_g) = \{f, g\}$. Hence,

$$\{f, gh\} = X_f(gh) = d(gh)(X_f) = (gdh + hdg)(X_f) = g\{f, h\} + h\{f, g\}$$

□

Lemma 4.1.25. $X_{\{f,g\}} = [X_f, X_g]$

Proof. We use $\iota_{[X,Y]} = \mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X$ to compute:

$$\iota_{[X_f,X_g]}\omega = \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathcal{L}_{X_f}\omega = \tag{4.1.21}$$

$$= d\circ\iota_{X_f}\circ\iota_{X_g}\omega + \iota_{X_f}\circ\underbrace{d\circ\iota_{X_g}\omega}_{=-d^2g=0} - \iota_{X_g}\circ\underbrace{d\circ\iota_{X_f}\omega}_{=-d^2f=0} - \iota_{X_g}\circ\iota_{X_f}\circ\underbrace{d\omega}_{=0} = \tag{4.1.22}$$

$$= d\circ\iota_{X_f}\circ\iota_{X_g}\omega = d\omega(X_g, X_f) = -d\omega(X_f, X_g) = -d\{f, g\} \tag{4.1.23}$$

□

Remark 4.1.26. We take the following two conventions:

1. $\omega(X_H, \cdot) = -dH$
2. $\iota_X\beta(Y_1, \dots, Y_{k-1}) = \beta(X, Y_1, \dots, Y_{k-1})$

Corollary 4.1.27. *$\Phi : f \longmapsto X_f : (\mathcal{C}^\infty(M), \{\cdot,\cdot\}) \longrightarrow (\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra morphism.*

Theorem 4.1.28. *Let $H \in \mathcal{C}^\infty$ be a Hamiltonian and X_H the corresponding Hamiltonian vector field. Then, $X_H(H) = 0$*

Proof. Consider the Poisson bracket $\{\cdot,\cdot\}$ defined by

$$\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\} = X_f(g) \tag{4.1.24}$$

where X_f is defined by $\iota_{X_f}\omega = -df$ and analogously for X_g. Then, by skew-symmetry of $\{\cdot,\cdot\}$, $X_H(H) = 0$. □

Definition 4.1.29. Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called symplectic if it preserves ω, i.e., if $\mathcal{L}_X\omega = 0$, or $d\iota_X\omega = 0$.

Remark 4.1.30. We will denote by $\mathfrak{X}^{\mathrm{Symp}}(M) = \{X \in \mathfrak{X}(M) \mid d\iota_X\omega = 0\}$ the set of symplectic vector fields on M and by $\mathfrak{X}^{\mathrm{Ham}}(M) = \{X \in \mathfrak{X}(M) \mid \iota_X\omega = -d\beta\}$ the set of symplectic vector fields on M.

Lemma 4.1.31. $\mathfrak{X}^{Ham}(M) \subset \mathfrak{X}^{Symp}(M)$.

Proof. Take $X \in \mathfrak{X}^{\mathrm{Ham}}(M)$. Then:

$$\mathcal{L}_{X_f}\omega = d\iota_{X_f}\omega + \iota_{X_f}d\omega = d(-df) + 0 = -d^2 f = 0 \tag{4.1.25}$$

□

Lemma 4.1.32. $\mathcal{H}^1(M) = {\mathfrak{X}^{Symp}(M)}\big/{\mathfrak{X}^{Ham}(M)}$

Example 4.1.33. Take $(\mathbb{R}^{2n}, \omega = dx \wedge dy)$. $X = \frac{\partial}{\partial x_1}$ is Hamiltonian because $\iota_X\omega = dy_1$. Hence, it is symplectic.

Example 4.1.34. Take $(S^2, \omega = dh \wedge d\theta)$. $X = \frac{\partial}{\partial \theta}$ is Hamiltonian because $\iota_X\omega = -dh$. Hence, it is symplectic.

Example 4.1.35. Take $(\mathbb{R}^{2n}, \omega = d\theta_1 \wedge \theta_2)$. $X = \frac{\partial}{\partial \theta_1}$ is not Hamiltonian because $\iota_X\omega = d\theta_2$ and θ_2 is not a global function. It is symplectic.

Lemma 4.1.36. *Let $f, H \in \mathcal{C}^\infty(M)$. Then,*

$$\{f, H\} = 0 \iff f \textit{ is constant along the flow of } X_H \tag{4.1.26}$$

Proof. $\{f, H\} = -X_H(f) = -\frac{d}{dt}(f \circ \gamma)(t)\big|_{t=0} = 0$ □

Definition 4.1.37. Suppose $f, H \in \mathcal{C}^\infty(M)$ satisfy $\{f, H\} = 0$. Then, f is called an *integral of motion of H*.

Definition 4.1.38. Let (M^{2n}, ω) be a symplectic manifold. A Hamiltonian system $(M, \omega, H \in \mathcal{C}^\infty(M))$ is called *completely integrable* if $\exists f_1, \dots, f_n \in \mathcal{C}^\infty(M)$ such that:

1. $\{f_i, f_j\} = 0$ for all $i, j = 1, \dots, n$
2. $df_1 \wedge \dots \wedge df_n \neq 0\, a.e.$

Example 4.1.39. Any Hamiltonian system defined on a surface, $(\Sigma^2, \omega, H \in \mathcal{C}^\infty(M))$, is completely integrable.

Example 4.1.40. The planar pendulum and the spherical pendulum are completely integrable systems.

Definition 4.1.41. Let G be a Lie group and (M, ω) a symplectic manifold. A group action $\rho : G \longrightarrow \mathrm{Diff}(M)$ is a *symplectic action* if $\rho : G \longrightarrow \mathrm{Symp}(M) \subset \mathrm{Diff}(M)$, where $\mathrm{Symp}(M)$ is the set of symplectomorphisms on M.

Example 4.1.42. Take $(\mathbb{R}^{2n}, \omega_{st})$ and $X = \partial/\partial x_1$. The flow of X defines a symplectic action $\psi : \mathbb{R} \longrightarrow \text{Symp}(\mathbb{R}^{2n}, \omega_{st})$ which is the following:

$$\psi(t)(x_1, y_1, \ldots, x_n, y_n) = (x_1 + t, y_1, \ldots, x_n, y_n) \tag{4.1.27}$$

Definition 4.1.43. A symplectic action ρ of S^1 or $\mathbb{R}$ on (M, ω) is called a *Hamiltonian action* if the field

$$X_x = \frac{d}{dt}\rho_t(x)\Big|_{t=0}$$

is Hamiltonian.

4.1.3 Revision of Lie Actions. Definitions

Definition 4.1.44. Let G be a Lie group. The following map:

$$\begin{array}{rccc} \psi : & G & \longrightarrow & \text{Diff}(G) \\ & g & \longmapsto & \psi(g) = \psi_g \end{array}$$

where $\psi_g(h) = ghg^{-1}$, the conjugation, is a Lie group action, the *conjugation action.*

Definition 4.1.45. The map $Ad_g = d(\psi_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$ induced a map

$$Ad : G \longrightarrow GL(\mathfrak{g})$$

, the *adjoint action.*

Definition 4.1.46. The map $Ad^* : G \longrightarrow GL(\mathfrak{g}^*)$ is the *coadjoint action.* It acts as:

$$Ad^*_g(X)(\xi) =< X, Ad^*_{g^{-1}}(\xi) > \tag{4.1.28}$$

Definition 4.1.47. A group action ρ of G on (M, ω) is Hamiltonian if $\exists\, \mu : M \longrightarrow \mathfrak{g}^*$ such that:

1. For every $X \in \mathfrak{g}$, $\{\exp(tX) \mid t \in \mathbb{R}\} \subset G$,

$$X^{\#}_x = \frac{d}{dt}(\exp(tX) \cdot x)\big|_{t=0},$$

$$\iota_{X^{\#}}\omega = -d\mu^X,$$

where $\mu^X(x) :=< \mu(x), X >$.

2. μ is equivariant with respect to the coadjoint action and the Lie group action, i.e.:
$$Ad_g^* \circ \mu = \mu \circ \rho_g$$
for every $g \in G$.

Then, (M, ω, G, μ) is called a *Hamiltonian G-space* and μ is called *moment map.*

Example 4.1.48. The Hamiltonian formulation of mechanical problems in Physics is one of the motivations of the development of symplectic geometry. Suppose $H \in \mathcal{C}^\infty(M^n)$ is the Hamiltonian (or energy function) of a system and assume that $H = H(q,p)$, where $(q,p) \in T^*M$, the cotangent bundle of M. In this setting, q is the positions vector and p is the momenta vector. Then, the system of ODEs that describe the evolution of positions and momenta is:
$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{cases} \tag{4.1.29}$$
Consider the 1-form $\alpha = \sum_{i=1}^n p_i dq_i \in \Omega^1(M)$, called the *Liouville 1-form.* Its differential is $d\alpha = \sum_{i=1}^n dp_i \wedge dq_i = \omega \in \Omega^2(M)$ and the Hamilton's equations (4.1.29) can be condensed in the expression $\iota_{X_H}\omega = -dH$.

4.2 Poisson Geometry

Definition 4.2.1. Let M be a smooth manifold. A $\mathcal{C}^\infty$ smooth *Poisson structure* is an $\mathbb{R}$-bilinear operation
$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & (f,g) & \longmapsto & \{f,g\} \end{array}$$
which satisfies:

1. Anti-symmetry, $\{f,g\} = -\{g,f\}$ for any $f,g \in \mathcal{C}^\infty(M)$
2. Leibniz rule, $\{f, g\cdot h\} = g \cdot \{f,h\} + \{f,g\} \cdot h$
3. Jacobi identity, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{g,\{f,g\}\} = 0$

The operation $\{\cdot,\cdot\}$ is also called *Poisson bracket.*

Example 4.2.2. Let M be any smooth manifold. Then, the operation
$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & (f,g) & \longmapsto & 0 \end{array}$$
is a Poisson structure because it automatically satisfies all of the required properties.

Remark 4.2.3. Example 4.2.2 shows that there are no topological obstructions on a manifold to admit a Poisson structure, i.e., any manifold can admit a Poisson structure.

Example 4.2.4. Let (M^{2n}, ω) be a symplectic manifold. Let X_H be Hamiltonian vector field of $H \in \mathcal{C}^\infty$, i.e., the solution of $\iota_{X_H}\omega = -dH$. Then, the operation

$$\begin{array}{rccc} \{\cdot,\cdot\}: & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & (f,g) & \longmapsto & \omega(X_f, X_g) \end{array}$$

is a Poisson structure on M. Since ω is 2-form, $\{f,g\}$ is bilinear. It is also anti-symmetric by anti-symmetry of ω. It satisfies Leibniz rule because

$$\{f,g\} = \omega(X_f, X_g) = \iota_{X_f}\omega(X_g) = \langle -df, X_g\rangle = -X_g(f) = X_f(g),$$

so $\{f,\cdot\}$ is $X_f(g)$, a derivation. Explicitly:

$$\{f, g\cdot h\} = \iota_{X_f}\omega(g\cdot h) = \iota_{X_f}\omega(g)\cdot h + g\cdot \iota_{X_f}\omega(h) = \{f,g\}\cdot h + g\cdot\{f,h\}.$$

It satisfies Jacobi identity because, since ω is closed, $d\omega(X_f, X_g, X_h) = 0$, and

$$\begin{aligned} 0 &= d\omega(X_f, X_g, X_h) = && (4.2.1)\\ &= X_f(\omega(X_g, X_h)) - X_g(\omega(X_f, X_h)) + X_h(\omega(X_f, X_g)) - && (4.2.2)\\ &\quad - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f) = && (4.2.3)\\ &= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\} - && (4.2.4)\\ &\quad - \{\{f,g\},h\} + \{\{f,h\},g\} - \{\{g,h\},f\} = && (4.2.5)\\ &= 2\left(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\right) && (4.2.6) \end{aligned}$$

So $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

Remark 4.2.5. According to Darboux Theorem (4.1.16), on any symplectic manifold M^{2n} there exist local coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ such that the symplectic form writes $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. In these coordinates, the Poisson bracket writes as

$$\{f,g\}^{\mathrm{st}} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right).$$

This is the formula that originally Siméon Denis Poisson presented in his article *Mémoire sur la variation des constantes arbitraires dans les questions de mécanique* in 1809.

Exercise 4.2.6. Consider $\mathbb{R}^2$ with the Poisson structure

$$\{f,g\} = H(x,y)\cdot\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

Proof that it is indeed a Poisson structure for any $H \in \mathcal{C}^\infty(\mathbb{R}^2)$.

Remark 4.2.7. In Exercise 4.2.6, if $H(x, y)$ is a constant function, then the Poisson structure is the standard Poisson structure of the symplectic case multiplied by a constant.

If $H(x, y)$ is regular (the differential does not vanish at any point), then in a neighbourhood of any point, i.e. locally, $H(x, y)$ can be written as $H(x, y) = x$ and the Poisson structure looks like

$$\{f, g\} = x \cdot \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right).$$

In this case, when $x \neq 0$ it gives a symplectic structure and when $x = 0$ it gives the structure in Example 4.2.2, the zero Poisson structure. This structure is called *b-symplectic structure* or *log-symplectic structure*.

Example 4.2.8. Consider $(S^2, \{\cdot, \cdot\})$, with coordinates h (for the height) and θ (for the angle) and with the Poisson structure defined as $\{f, g\} = h \cdot \{f, g\}^{\mathrm{st}}$. This structure is symplectic on the North hemisphere ($h > 0$) and also on the South hemisphere ($h < 0$), while all the points in the equator are symplectic leaves (of dimension 0).

If $H(x, y)$ has singularities, the Poisson structure can locally have many other forms. For instance, the so-called *c-symplectic structure* is the one defined by

$$\{f, g\} = x \cdot y \cdot \{f, g\}^{\mathrm{st}}.$$

4.2.1 Local Coordinates

We assume in this section that we have a local coordinate system $(x_1, \ldots, x_n)$ in a Poisson manifold and we denote by ω_{ij} the Poisson bracket of the coordinate functions x_i, x_j, i.e. $\omega_{ij} = \{x_i, x_j\}$.

Proposition 4.2.9. *The functions ω_{ij} have the following properties:*

1. $\omega_{ij} = -\omega_{ji}$, *for any* i, j.
2. $\{\omega_{ij}, x_k\} + \{\omega_{jk}, x_i\} + \{\omega_{ki}, x_j\} = 0$, *for any* i, j, k.
3. $\sum_{l=1}^{n} \left(\omega_{li}\frac{\partial \omega_{jk}}{\partial x_l} + \omega_{lj}\frac{\partial \omega_{ki}}{\partial x_l} + \omega_{lk}\frac{\partial \omega_{ij}}{\partial x_l} \right)$, *for any* i, j, k.

Proposition 4.2.10. *Using Property 3 in Proposition 4.2.9, the Poisson bracket can be computed as:*

$$\{f, g\} = [df]^T \cdot \omega \cdot [dg],$$

where $\omega = (\omega_{ij})_{ij}$.

Conversely, given a skew-symmetric matrix $\omega = (\omega_{ij})_{ij}$, the formula

$$\{f, g\} = \sum_{i,j=1}^{n} \omega_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

defines a Poisson bracket if the ω_{ij} satisfy Property 3 in Proposition 4.2.9.

Example 4.2.11. Suppose that ω_{ij} are linear functions for any i, j and assume that they have the form $\omega_{ij} = \sum_{k=1}^{n} c_{ij}^{k} x_k$, with c_{ij}^{k} constants. Then, we have the following equivalence:

$$\omega_{ij} \text{ satisfy Property 3 in Prop 4.2.9} \iff c_{ij}^{k} \text{ satisfy Jacobi identity,} \tag{4.2.7}$$

which is the same as to say that the c_{ij}^{k} are the structural constants of the dual of a Lie subalgebra of $\mathbb{R}^n$.

Example 4.2.12. Suppose that ω_{ij} are constant functions for any i, j. The Poisson structure in this case is called a *regular Poisson structure*, and the matrix $\omega = (\omega_{ij})_{ij}$ is of constant rank.

4.2.2 Bivector fields

Definition 4.2.13. Let M be a smooth manifold. A bivector field Π on M is a section of the two exterior power of the tangent bundle, i.e., $\Pi = \Gamma(\bigwedge^2 TM)$. In local coordinates:

$$\Pi = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

Proposition 4.2.14. *Let $(M, \{\cdot, \cdot\})$ be a Poisson structure in a smooth manifold. Then, there exists a bivector field Π such that, for any pair of functions $f, g \in \mathcal{C}^\infty(M)$, $\{f, g\} = \Pi(df, dg)$.*

Proof. The coefficients Π_{ij} of the bivector field Π are defined locally by:

$$\Pi_{ij} := \omega_{ij} = \{x_i, x_j\} \tag{4.2.8}$$

and the equality $\{f, g\} = \Pi(df, dg)$ follows from the definition and bilinearity of $\{\cdot, \cdot\}$. □

Proposition 4.2.15. *Given a bivector field Π, the operator $\{f, g\}_\Pi := \Pi(df, dg)$ satisfies:*

1. *Anti-symmetry*
2. *$\mathbb{R}$-bilinearity*

3. *Leibniz rule*

Proof. Anti-symmetry and bilinearity are automatic from definition of a bivector field Π. Since $\{f, \cdot\} = \Pi(df, \cdot)$ acts as a derivation, it also satisfies Leibniz rule. $\square$

Remark 4.2.16. The operator $\{f, g\}_\Pi := \Pi(df, dg)$ does not satisfy Jacobi identity in general. In Example 4.2.17, for instance, Jacobi identity fails. In consequence, not any bivector field gives raise to a Poisson structure. If Π satisfies $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket (see Definition 4.2.18), then it satisfies Jacobi identity and it does give raise to a Poisson structure. From now on, if (M, Π) is said to be a Poisson structure, it will be assumed that $[\Pi, \Pi] = 0$.

Example 4.2.17. Let $M = \mathbb{R}^3$ and take $\Pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$. In this case, by definition, $\{x, y\} = 1$ and $\{x, z\} = x$. Then:

$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = \tag{4.2.9}$$
$$= \{x, 0\} + \{y, -x\} + \{z, 1\} = \tag{4.2.10}$$
$$= 0 + \{x, y\} + 0 = \tag{4.2.11}$$
$$= 1 \neq 0 \tag{4.2.12}$$

meaning that the operator $\{f, g\}_\Pi := \Pi(df, dg)$ with the bivector field $\Pi = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$ does not satisfy Jacobi identity.

The Schouten Bracket

Definition 4.2.18. The *Schouten bracket* is the extension of the Lie bracket of vector fields to alternating multi-vector fields. If $A = a_1 \wedge \cdots \wedge a_n$ and $B = b_1 \wedge \cdots \wedge b_m$ are two multi-vector fields of degree n and m respectively, then the Schouten Bracket between A and B is defined in terms of the Lie bracket of vector fields $[a_i, b_j]$ by:

$$[A, B] = \sum_{i,j} (-1)^{i+j} [a_i, b_j] a_1 \cdots a_{i-1} a_{i+1} \cdots a_m b_1 \cdots b_{j-1} b_{j+1} \cdots b_n, \tag{4.2.13}$$

where all the products are wedge products.

Exercise 4.2.19. Let $A = a_1 \wedge \cdots \wedge a_a$, $B = b_1 \wedge \cdots \wedge b_b$ and $C = c_1 \wedge \cdots \wedge c_c$ be two multi-vector fields of degree a, b and c respectively. Proof that the Schouten bracket satisfies the following properties:

1. Graded anti-symmetry:

$$[A, B] = -(-1)^{(a-1)(b-1)} [B, A]$$

2. Graded Leibniz rule:

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$$

3. Graded Jacobi identity:

$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0$$

Exercise 4.2.20. Proof that the Schouten bracket satisfies $[\Pi, \Pi] = 0$ for $\Pi = y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$.

4.2.3 The Poisson category

Definition 4.2.21. If $(M, \{\cdot, \cdot\}_1)$ and $(N, \{\cdot, \cdot\}_2)$ are two Poisson structures on the manifolds M and N, a mapping $\phi : (M, \{\cdot, \cdot\}_1) \longrightarrow (N, \{\cdot, \cdot\}_2)$ is called a *Poisson morphism* if $\phi^*(\{f, g\}_2) = \{\phi^* f, \phi^* g\}_1$ for any $f, g \in \mathcal{C}^\infty(N)$.

Example 4.2.22. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and $\mathfrak{g}^*$ the dual of a Lie algebra. Then, a moment map $F : (M, \{\cdot, \cdot\}) \longrightarrow \mathfrak{g}^*$ is a Poisson morphism.

Example 4.2.23. Consider a Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h}$ and the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. The dual of the inclusion, $i^* : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ is a Poisson morphism.

Definition 4.2.24. Consider a Poisson manifold (M, Π_M) and a submanifold (N, Π_N) which also has a Poisson structure. (N, Π_N) is a *Poisson submanifold* if the inclusion map $(N, \Pi_N) \overset{i}{\hookrightarrow} (M, \Pi_M)$ is a Poisson morphism.

Definition 4.2.25. Let (M, Π) be a Poisson manifold. A *Poisson vector field* is a vector field X such that $\mathcal{L}_X \Pi = 0$.

Remark 4.2.26. If X is a Poisson vector field, φ_t^X, the flow of X, is a Poisson morphism.

Remark 4.2.27. The following equivalence holds and gives an alternative definition of a Poisson vector field:

$$\mathcal{L}_X \Pi = 0 \iff X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\} \tag{4.2.14}$$

Definition 4.2.28. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. A *Hamiltonian vector field associated to a function* $f \in \mathcal{C}^\infty(M)$ is defined by $X_f = \{f, \cdot\}$.

Remark 4.2.29. In notation of vector fields, $X_f = \Pi(df, \cdot)$.

Proposition 4.2.30. *Suppose X_f is a Hamiltonian vector field. Then:*

1. *X_f is a Poisson vector field.*

2. $X_f(f) = 0$.

3. If X_g *is also Hamiltonian,* $[X_f, X_g] = X_{\{f,g\}}$

Proof. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Suppose X_f is the Hamiltonian vector field associated to f.

1. $X_f(\{g,h\}) = \{f,\{g,h\}\} = \{g,\{f,h\}\} + \{\{f,g\},h\} =$
   $= \{g, X_f(h)\} + \{X_f(g), h\}$ for any $g,h \in \mathcal{C}^\infty(M)$, so X_f is Poisson.
2. $X_f(f) = \{f,f\} = 0$ by anti-symmetry.
3. See the proof of Lemma 4.1.25.

□

4.2.4 Symplectic foliations, splitting theorem and normal forms

Theorem 4.2.31 (Stefan-Sussmann Theorem)**.** *Let M be a smooth manifold and let $\mathcal{D}$ be a smooth distribution. Then, $\mathcal{D}$ is integrable if and only if it is generated by a family C of smooth vector fields, and is invariant with respect to C.*

Consider a Poisson manifold $(M, \{\cdot,\cdot\})$ and the set $\mathcal{D} = \{X_f \mid f \in \mathcal{C}^\infty(M)\}$ of Hamiltonian vector fields, which is, in fact, a distribution.

Proposition 4.2.32. *The distribution $\mathcal{D} = \{X_f \mid f \in \mathcal{C}^\infty(M)\}$ satisfies the conditions of Stefan-Sussmann and, therefore, there exists a (singular) foliation $\mathcal{F}$ integrating $\mathcal{D}$ which is symplectic.*

Theorem 4.2.33 (Weinstein Splitting Theorem, '83)**.** *In a neighbourhood of a point p in a Poisson manifold $(M, \{\cdot,\cdot\})$ there exist local coordinates centered at p $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_l)$ such that:*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{l} \varphi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \tag{4.2.15}$$

and with $\varphi_{ij}(0) = 0$.

Remark 4.2.34. Weinstein Splitting Theorem tells us that, at each point $p \in M$, the Poisson structure splits in two parts. In a neighbourhood of p, the Poisson structure Π is the sum of $\Pi_S = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$, a part which is dual to the symplectic structure of the leaf through p, and $\Pi_T \sum_{i,j=1}^l \varphi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. The condition $[\Pi, \Pi] = 0$ on the whole manifold implies that Π_T is a Poisson structure by itself and it is called the *transverse Poisson structure.*

Remark 4.2.35. This transverse Poisson structure does not depend on the particular transversal, it only depends on the symplectic leaf.

Remark 4.2.36. Weinstein Splitting Theorem is the closest Poisson analogous to Darboux Theorem (4.1.16) for symplectic manifolds.

Remark 4.2.37. The rank of the Poisson structure Π at p is $2k$ and it equals the dimension of the symplectic leaf at p.

Definition 4.2.38. Let (M, Π) be a Poisson manifold. The *anchor map* is the following map:

$$\begin{array}{cccc} \Pi^{\#}: & T^*M & \longrightarrow & TM \\ & \alpha & \longmapsto & \Pi(\alpha, \cdot) \end{array}$$

Remark 4.2.39. Recall that $\mathcal{D}_x = \{X_f \mid f \in \mathcal{C}^\infty(M)\}$ is the distribution tangent to the symplectic foliation at a point $x \in M$. This distribution $\mathcal{D}_x$ is precisely the image of $\Pi^{\#}(x)$ and the rank of Π at x equals $\Pi^{\#}_x$, the dimension of the symplectic leaf through x.

Theorem 4.2.40 (Conn, Ginzburg)**.** *Consider a Poisson manifold* (M, Π) *and the splitting* $\Pi = \Pi_S + \Pi_T$ *given by Theorem 4.2.33. If the linear part of* Π_T *is semisimple of compact type, then we can write, locally:*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j,k=1}^{l} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}. \tag{4.2.16}$$

Remark 4.2.41. The linearization of the transverse structure, $\Pi_T^{(l)}$, corresponds to the dual of a Lie algebra of semisimple compact type.

Remark 4.2.42. It is necessary to ask the linear part to be of compact type. To show it, take the following counterexample of non-compact type: Consider $\mathfrak{SL}(2, \mathbb{R})^*$, which is a semisimple algebra but is not of compact type. There is a way to perturb $\mathfrak{SL}(2, \mathbb{R})^*$ in such a way that it is not equivalent to the linear model (see [Wei83]).

4.2.5 Poisson Cohomology

The definition of the Schouten bracket (4.2.18) shows that the bracket of a multi-vector field A of degree a with a multi-vector field B of degree b produces a multi-vector field of degree $a+b-1$. If B is a bivector field, then the Schouten Bracket of any multi-vector field A with B increases the degree of A in one. Then, it is a good candidate for the differential of a Poisson Cohomology.

Lemma 4.2.43. *Let (M,Π) be Poisson structure. Given A a multi-vector field of any degree a, the following equation holds:*

$$[\Pi,[\Pi,A]] = 0 \tag{4.2.17}$$

Proof. By the graded Jacobi identity applied to the Schouten bracket of A of degree a and Π of degree 2, and using that $[\Pi,\Pi]=0$, we obtain:

$$(-1)^{a-1}[\Pi,[\Pi,A]] - [\Pi,[A,\Pi]] = 0 \tag{4.2.18}$$

By the graded anti-symmetry of the Schouten bracket:

$$[A,\Pi] = -(-1)^{a-1}[\Pi,A] = (-1)^{a}[\Pi,A] \tag{4.2.19}$$

So, combining 4.2.18 and 4.2.19:

$$\begin{aligned} 0 =&(-1)^{a-1}[\Pi,[\Pi,A]] - [\Pi,[A,\Pi]] = && (4.2.20)\\ =&(-1)^{a-1}[\Pi,[\Pi,A]] - (-1)^{a}[\Pi,[\Pi,A]] = && (4.2.21)\\ =&\left((-1)^{a-1} + (-1)^{a-1}\right)[\Pi,[\Pi,A]] = && (4.2.22)\\ =&2(-1)^{a-1}[\Pi,[\Pi,A]] && (4.2.23)\end{aligned}$$

□

Definition 4.2.44. Let (M,Π) be a Poisson manifold. Consider multi-vector fields of degree k, i.e., elements of $\mathfrak{X}^k(M)$, we can construct the following chain:

$$\cdots \xrightarrow{d_\Pi} \mathfrak{X}^{k-1}(M) \xrightarrow{d_\Pi} \mathfrak{X}^{k}(M) \xrightarrow{d_\Pi} \mathfrak{X}^{k+1}(M) \xrightarrow{d_\Pi} \cdots \tag{4.2.24}$$

where $d_\Pi(A) = [\Pi,A]$. Because of Lemma 4.2.43, $d_\Pi^2 = [\Pi,[\Pi,A]] = 0$. Therefore, we can define the *Poisson Cohomology groups* $H^k_\Pi(M)$ as:

$$H^k_\Pi(M) = {\ker\left(d_\Pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)\right)}\Big/{\mathrm{Im}\left(d_\Pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^{k}(M)\right)} \tag{4.2.25}$$

Example 4.2.45. In a manifold (M,Π) with the zero Poisson structure, i.e., with $\Pi = 0$, the differential is $d_\Pi = 0$. Then, all the cohomology groups of the Poisson cohomology are the multi-vector fields, $H^k_\Pi(M) = \mathfrak{X}^k(M)$. It is an infinite-dimensional cohomology.

Example 4.2.46. In a manifold (M,ω) with a symplectic structure, the Poisson cohomology groups are isomorphic to the De Rham cohomology groups, $H^k_\Pi(M) \cong H^k_{DR}(M)$.

This holds because the anchor map $\Pi^{\#} : T^*M \to TM$ (see Definition 4.2.38) is an isomorphism. The the wedge product keeps the isomorphy and, therefore, we can compare both cohomologies because $\Pi^{\#}$ induces a map from forms to multi-vector fields such that this diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_{DR}} & \Omega^{k-1}(M) & \xrightarrow{d_{DR}} & \Omega^{k}(M) & \xrightarrow{d_{DR}} & \Omega^{k+1}(M) & \xrightarrow{d_{DR}} & \cdots \\ & & \big\downarrow \Pi^{\#} & & \big\downarrow \Pi^{\#} & & \big\downarrow \Pi^{\#} & & \\ \cdots & \xrightarrow{d_{\Pi}} & \bigwedge^{k-1}(M) & \xrightarrow{d_{\Pi}} & \bigwedge^{k}(M) & \xrightarrow{d_{\Pi}} & \bigwedge^{k+1}(M) & \xrightarrow{d_{\Pi}} & \cdots \end{array} \tag{4.2.26}$$

Example 4.2.47. Consider a b-symplectic manifold, i.e., a Poisson manifold (M^{2n}, Π) of even dimension such that the map

$$\begin{array}{rccc} \Pi^n : & M^{2n} & \longrightarrow & \bigwedge^{2n}(TM) \\ & x & \longmapsto & \Pi^n(x) \end{array}$$

is transverse to the zero section of the fiber bundle of the manifold. In this manifold, locally Π writes in the following way:

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} \tag{4.2.27}$$

And there is the following result.

Theorem 4.2.48. *Let M be a compact manifold. Then, the following holds:*

$$H^k_{\Pi}(M) \cong H^k_{DR}(M) \oplus H^{k-1}_{DR}(Z) \tag{4.2.28}$$

where $Z = \{p \in M \mid \Pi^n(p) = 0\}$, a codimension 1 submanifold of M which is intrinsically associated to the b-Poisson structure.

So the cohomology in the b-symplectic case is almost like in the symplectic case but with an additional term.

Example 4.2.49. If $\mathfrak{g}$ is a Lie algebra and U neighbourhood a representation. In general, $H^k_{\Pi}(U) \cong H^*(\mathfrak{g}, \mathcal{C}^\infty(U)) \otimes C^\infty_{\text{Ind}}(M)$. We use the Chevalley Eilenberg Cohomology, which is associated to the representation of the Lie algebra on the space $\mathcal{C}^\infty(M)$ given by the Poisson bracket $\{\cdot, \cdot\}$.

In the case that $\mathfrak{g}$ a semisimple Lie algebra of compact type. The Poisson cohomology of $\mathfrak{g}$ a is:

$$H^*_{\Pi}(U) \cong H^*(\mathfrak{g}) \otimes (\mathcal{C}^\infty(U))^G .$$

Interpretation of the Poisson cohomology groups

Assume (M, Π) is a Poisson structure. The cohomology group $H^0_\Pi(M)$ is $\{f \in \mathcal{C}^\infty(M) \mid X_f = 0\}$ the set of functions such that the Hamiltonian vector field is zero. It is called the set of *Casimir functions* and can be also written as $\{f \in \mathcal{C}^\infty(M) \mid \{f, g\} = 0 \,\forall g \in \mathcal{C}^\infty(M)\}$.

The cohomology group $H^1_\Pi(M)$ is the set of Poisson vector fields quotiented by the set of Hamiltonian vector fields, because, $\ker([\Pi, A])$ for $A \in \mathfrak{X}(M) = \mathcal{C}^\infty(M)$ is exactly the set of Poisson vector fields and $\operatorname{Im}([\Pi, f])$ for $f \in \mathfrak{X}^0(M) = \mathcal{C}^\infty(M)$ is precisely the set of Hamiltonian vector fields.

The cohomology group $H^2_\Pi(M)$ is, explicitly:

$$H^2_\Pi(M) = {}^{\{\Pi \mid [\Pi,\Pi]=0\}}\big/_{\{\Pi \mid \Pi = [\Pi, X]\,\forall X \in \mathfrak{X}(M)\}}$$

It can be interpreted in terms of first order infinitesimal deformations modulo trivial deformations.

Up to this point, it is possible to make the following analogies between Symplectic and Poisson structures:

Symplectic Structure	Poisson Structure
$\omega \in \Omega^2(M)$	$\Pi \in \mathfrak{X}^2(M)$
$d\omega = 0$	$[\Pi, \Pi] = 0$
$\iota_{X_f}\omega = -df$	$X_f = \Pi(df, \cdot)$
$\mathcal{L}_X\omega = 0$	$\mathcal{L}_X\Pi = 0$
$H^k_{DR}(M)$	$H^k_\Pi(M)$

Definition 4.2.50. Two Poisson structures Π_1, Π_2 on a manifold M are *compatible* if $[\Pi_1, \Pi_2] = 0$.

Remark 4.2.51. If Π_1 and Π_2 are compatible, then Π_2 is a 2-cocycle in $H^2_{\Pi_1}(M)$ and vice-versa.

Remark 4.2.52. If Π_1, Π_2 are two Poisson structures on a manifold M, then $\alpha\Pi_1 + \beta\Pi_2$, with α, β constants, is a Poisson structure if and only if Π_1 and Π_2 are compatible. This is because $[\alpha\Pi_1 + \beta\Pi_2, \alpha\Pi_1 + \beta\Pi_2] = 2\alpha\beta[\Pi_1, \Pi_2]$ since $[\Pi_1, \Pi_1] = [\Pi_2, \Pi_2] = 0$.

Definition 4.2.53. A vector field X is called *bi-Hamiltonian* with respect to two compatible Poisson structures Π_1 and Π_2 if X is at the same time Hamiltonian with respect to Π_1 and with respect to Π_2, i.e., if $X = X_{f_1}^{\Pi_1} = X_{f_2}^{\Pi_2}$.

Proposition 4.2.54. *Let X be bi-Hamiltonian and let f_1, f_2 be its Hamiltonian functions. Then, $\{f_1, f_2\}_{\Pi_1} = \{f_1, f_2\}_{\Pi_2} = 0$.*

Proof. We know that $\{f_1, f_1\}_{\Pi_1} = 0$, by anti-symmetry. Then:

$$0 = \{f_1, f_1\}_{\Pi_1} = X_{f_1}^{\Pi_1}(f_1) = X_{f_2}^{\Pi_2}(f_1) = \{f_2, f_1\}_{\Pi_2} \tag{4.2.29}$$

Changing the role of f_1 and f_2 the equality $\{f_1, f_2\}_{\Pi_1}$ is also proved. □

Example 4.2.55. Take a holomorphic function $F : \mathbb{C}^2 \to \mathbb{C}$ and decompose it as $F = G + iH$ with $G, H : \mathbb{R}^2 \mapsto \mathbb{R}$. The Cauchy-Riemann equations for F in coordinates $z_i = x_i + y_i$ are:

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i} \ , \ \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i} \tag{4.2.30}$$

and imply that there is a vector field which is bi-Hamiltonian with respect to two Poisson structures (in this case symplectic structures) which are compatible. Then, we can rewrite the Cauchy-Riemann equations as:

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \ , \ \{G, \cdot\}_1 = -\{H, \cdot\}_0 \tag{4.2.31}$$

4.2.6 Integrable Systems

If we decompose the symplectic form in $\omega = dz_1 \wedge dz_2 = \omega_0 + i\omega_1$, the bracket $\{\cdot, \cdot\}_0$ corresponds to the Poisson bracket associated to ω_0 and the bracket $\{\cdot, \cdot\}_1$ corresponds to the Poisson bracket associated to ω_1. This implies automatically that the real and imaginary parts of a holomorphic function provide commuting functions for both Poisson structures.

Definition 4.2.56. Let (M, ω) be a symplectic manifold. An *integrable system* on M is given by n functions $f_1, \dots, f_n$ such that:

1. $f_1, \dots, f_n$ are independent in a dense set.
2. $\{f_i, f_j\} = 0$ for any i, j, where $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j})$.

Remark 4.2.57. The functions $f_1, \dots, f_n$ are independent in a dense set if and only if $df_1 \wedge \dots \wedge df_n \neq 0$ in a dense set.

Exercise 4.2.58. Given two functions $H, K \in \mathcal{C}^\infty(\mathbb{R}^3)$, consider the system of differential equations given by:

$$(\dot{x}, \dot{y}, \dot{z}) = dH \wedge dK \tag{4.2.32}$$

The field $dH \wedge dK$ is a bi-Hamiltonian vector field. We define the following two Poisson structures:

$$\begin{array}{rccccl} \{\cdot,\cdot\}_H : & \mathcal{C}^\infty(\mathbb{R}^3) \times \mathcal{C}^\infty(\mathbb{R}^3) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^3) & & \\ & (f,g) & \longmapsto & \det(df, dg, dH) & =: \{f,g\}_H \end{array}$$

$$\begin{array}{rccccl} \{\cdot,\cdot\}_K : & \mathcal{C}^\infty(\mathbb{R}^3) \times \mathcal{C}^\infty(\mathbb{R}^3) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R}^3) & & \\ & (f,g) & \longmapsto & \det(df, dg, dK) & =: \{f,g\}_K \end{array}$$

Proof that the flow of the vector field $\{K, \cdot\}_H := \det(dK, \cdot, dH)$, which is the same as $\{-H, \cdot\}_K$, is the solution of 4.2.32 and it is bi-Hamiltonian with respect to $\{\cdot,\cdot\}_H$ and $\{\cdot,\cdot\}_K$.

Example 4.2.59. Consider a surface and suppose an integrable system is defined on it. Since it is a 2-dimensional manifold, the integrable system is given by a single function μ. If we assume that the fibers are compact, then they can only be circles or points and they form a fibration.

If the surface is a 2-sphere and the function is the height function h, the dynamics of the system is given by rotations around the height axis. Explicitly, if we consider the symplectic setting and we take $\omega = dh \wedge d\theta$, the dynamical system is given by $\iota_{\frac{\partial}{\partial\theta}}\omega = -dh$. It is a Hamiltonian system and h is a moment map.

Definition 4.2.60. Let (M, Π) be a Poisson manifold of dimension n and of (maximal) rank $2r$. A family of functions $f_1, \dots, f_s \in \mathcal{C}^\infty(M)$ defines a *Liouville integrable system* on (M, Π) if:

1. $f_1, \dots, f_s$ are independent almost everywhere (i.e., their differentials are independent on a dense open subset of M),
2. $f_1, \dots, f_s$ are pairwise in involution,
3. $r + s = n$.

Viewed as a map, $F := (f_1, \dots, f_s) : M \mapsto \mathbb{R}^s$ is called the *momentum map* of the system.

For a given Liouville integrable system, we have two different foliations. First, we can consider the span of the set of Hamiltonian vector fields corresponding to the s components of the moment map F, i.e., $D = \langle X_{f_1}, \dots, X_{f_s} \rangle$.

It is indeed an integrable distribution because $[X_{f_i}, X_{f_j}] = X_{\{f_i,f_j\}} = X_0 = 0$ for every i, j, since $\{f_i, f_j\} = 0$ for every i, j because they are in involution. Then, there exists a foliation $\mathcal{F}^\infty$ given by the integral manifolds of this distribution. At each point $m \in M$, it satisfies $T_m(\mathcal{F}^1_m) = D_m$, where $\mathcal{F}^1_m$ is the leaf of $\mathcal{F}^1$ through the point m. The leafs of $\mathcal{F}^1$ are called *invariant submanifolds*.

We can also consider the moment map $F = (f_1, \dots, f_s)$, which defines a fibration on M. Then, it defines a second foliation on M, $\mathcal{F}^2$. In a regular point $m \in M$, it coincides with $\mathcal{F}^1$.

Theorem 4.2.61. *Let m be a point of a Poisson manifold (M, Π) of dimension n. Let $p_1, \dots, p_r$ be functions in involution, defined on a neighborhood of m, which vanish at m and whose Hamiltonian vector fields are linearly independent at m. Then there exist, on a neighbourhood U of m, functions $q_1, \dots, q_r, z_1, \dots, z_{n-2r}$, such that:*

1. *The n functions $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_{n-2r})$ form a system of coordinates on U, centered at m,*
2. *The Poisson structure Π is given on U by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4.2.33}$$

where each function $g_{ij}(z)$ is a smooth function on U and is independent of $p_1, \dots, p_r, q_1, \dots, q_r$.

Remark 4.2.62. The rank of Π at m is $2r$ if and only if all the functions $g_{ij}(z)$ vanish for $z = 0$.

Remark 4.2.63. In general, integrable systems on Poisson manifolds do not split in the sense of Weinstein Splitting Theorem (4.2.33), i.e., it is not possible to separate F into $F = (F_S, F_T)$.

In a 2010 paper of Laurent-Gengoux, Miranda, Vanhaecke [LMV11], an action-angle theorem in the general context 4.2.64 of integrable systems on Poisson manifolds is proved, as well as for the version for non-commutative integrable systems 4.2.65.

Theorem 4.2.64. *Let (M, Π, F) be an integrable system, where (M, Π) is a Poisson manifold of dimension n and rank $2r$ and F is a moment map. Suppose that, at a point $m \in M$, $\mathcal{F}_m$ is a standard Liouville torus. Then, there exist $\mathbb{R}$-valued smooth functions $(p_1, \dots, p_{n-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \dots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$, such that:*

1. *The functions* $(\theta_1, \dots, \theta_r, p_1, \dots, p_{n-r})$ *define an isomorphism* $U \simeq \mathbb{T}^r \times B^{n-r}$*,*

2. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i}.$$

   In particular, the functions $p_{r+1}, \dots, p_{n-r}$ *are Casimirs of* Π *(restricted to* U*),*

3. *The leaves of the surjective submersion* $F = (f_1, \dots, f_{n-r})$ *are given by the projection onto the second component* $\mathbb{T}^r \times B^{n-r}$*. In particular, the functions* $p_1, \dots, p_{n-r}$ *depend only on the functions* $f_1, \dots, f_{n-r}$*.*

The functions $\theta_1, \dots, \theta_r$ *are called angle coordinates, the functions* $p_1, \dots, p_r$ *are called action coordinates and the remaining coordinates* $p_{r+1}, \dots, p_{n-r}$ *are called transverse coordinates.*

Proof. We denote $s := n - r$. Since $\mathcal{F}_m$ is a standard Liouville torus, on a neighborhood U' of $\mathcal{F}_m$ in M there exist, on the one hand, Casimir functions $p_{r+1}, \dots, p_s$. On the other hand, there exist F-basic functions $p_1, \dots, p_r$ such that $p := (p_1, \dots, p_s)$ and F define the same foliation on U' and such that the Hamiltonian vector fields $X_{p_1}, \dots, X_{p_r}$ are the fundamental vector fields of a $\mathbb{T}^r$-action on U', where each of the vector fields has period 1. The orbits of this torus action are the leaves of the latter foliation. In view of the Theorem (theorem 4.2.61), on a neighborhood $U'' \subset U'$ of m in M, there exist $\mathbb{R}$-valued functions $\theta_1, \dots, \theta_r$ such that:

$$\Pi = \sum_{j=1}^{r} \frac{\partial}{\partial \theta_j} \wedge \frac{\partial}{\partial p_j}. \tag{4.2.34}$$

On U'', $X_{p_j} = \frac{\partial}{\partial \theta_j}$, for $j = 1, \dots, r$. Since each of these vector fields has period 1 on U', it is natural to view these functions as $\mathbb{R}/\mathbb{Z}$-valued functions, which we do without changing the notation. Notice that the functions $\theta_1, \dots, \theta_r$ are independent and pairwise in involution on U'', as a trivial consequence of Equation 4.2.34. In particular, $\theta_1, \dots, \theta_r, p_1, \dots, p_s$ define local coordinates on U''. In these coordinates, the action of $\mathbb{T}^r$ is given by:

$$(t_1, \dots, t_r) \cdot (\theta_1, \dots, \theta_r, p_1, \dots, p_s) = (\theta_1 + t_1, \dots, \theta_r + t_r, p_1, \dots, p_s), \tag{4.2.35}$$

so that the functions θ_i uniquely extend to smooth $\mathbb{R}/\mathbb{Z}$-valued functions satisfying (4.2.35) on $U := F^{-1}(F(U''))$, which is an open subset of $\mathcal{F}_m$ in M. The extended functions are still denoted by θ_i. It is clear that $\{\theta_i, p_j\} = \delta_i^j$ on U, for all $i, j = 1, \ldots, r$. Combined with the Jacobi identity, this leads to

$$X_{p_k}[\{\theta_i, \theta_j\}] = \{\{\theta_i, \theta_j\}, p_k\} = \{\theta_i, \delta_j^k\} - \{\theta_j, \delta_i^k\} = 0,$$

which shows that the Poisson brackets $\{\theta_i, \theta_j\}$ are invariant under the $\mathbb{T}$-action. But the latter vanish on U'', hence these brackets vanish on all of U, and we may conclude that on U, the functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_s)$ have independent differentials, so they define a diffeomorphism to $\mathbb{T}^r \times B^s$ where B^s is a (small) ball with center 0, and that the Poisson structure takes in terms of these coordinates the canonical form (4.2.34), as required. □

The action-angle theorem can be extended to standard Liouville tori of a non-commutative integrable system.

Theorem 4.2.65. *Let (M, Π) be a Poisson manifold of dimension n, equipped with a non-commutative integrable system $F = (f_1, \ldots, f_s)$ of rank r. Suppose that, at a point $m \in M$, $\mathcal{F}_m$ is a standard Liouville torus. Then, there exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$, such that*

1. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define an isomorphism $U \simeq \mathbb{T}^r \times B^s$,*
2. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{k,l}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l},$$

3. *The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$. In particular, the functions $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ depend only on the functions $f_1, \ldots, f_s$.*

The functions $\theta_1, \ldots, \theta_r$ are called angle coordinates, the functions $p_1, \ldots, p_r$ are called action coordinates and the remaining coordinates $z_1, \ldots, z_{s-r}$ are called transverse coordinates.

Proof. Conditions (1) and (2) imply that, on a neighborhood U' of $\mathcal{F}_m$ in M, there exist, on the one hand, F-basic functions $z_1, \ldots, z_{s-r}$ and, on the other hand, Cas-basic functions $p_1, \ldots, p_r$, such that:

- $$p := (p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$$

  and F define the same foliation on U',

- The Hamiltonian vector fields $X_{p_1}, \ldots, X_{p_r}$ are the fundamental vector fields of a $\mathbb{T}^r$-action on U', where each has period 1.

The orbits of this torus action are the leaves of the latter foliation. In view of Theorem 4.2.61, on a neighborhood $U'' \subset U'$ of m in M there exist $\mathbb{R}$-valued functions $\theta_1, \ldots, \theta_r$ such that

$$\Pi = \sum_{j=1}^{r} \frac{\partial}{\partial \theta_j} \wedge \frac{\partial}{\partial p_j} + \sum_{k,l=1}^{s-r} \phi_{k,l}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}. \tag{4.2.36}$$

The end of the proof goes along the same lines as the end of the proof of theorem 4.2.64. $\square$

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