$b^m$-Contact Geometry: Periodic Orbits and the 3BP

Cédric Oms

Universidad Politécnica de Catalunya

VLAB meeting

Día 57 del confinamiento 2020
Motivating examples from celestial mechanics
Restricted planar circular 3-body problem I

Simplified version of the general 3-body problem:

- One of the bodies has negligible mass.
- The other two bodies move in circles following Kepler’s laws for the 2-body problem.
- The motion of the small body is in the same plane.
Restricted planar circular 3-body problem II

- Time-dependent potential: $U(q, t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
- Time-dependent Hamiltonian:
  $$H(q, p, t) = \frac{|p|^2}{2} - U(q, t), \quad (q, p) \in \mathbb{R}^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$$
- Rotating coordinates: Time independent Hamiltonian
  $$H(q, p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1q_2 - p_2q_1$$
- $H$ has 5 critical points: $L_i$ Lagrange points ($H(L_1) \leq \cdots \leq H(L_5)$)
- Periodic orbits of $X_H$?
- Perturbative methods (dynamical systems) or.... contact geometry!
Level-sets of Hamiltonians

Let \((\mathcal{W}, \omega)\) be a symplectic manifold and \(\Sigma \subset \mathcal{W}\) hypersurface.

**Definition**

A Liouville vector field is a v.f. \(X \in \mathfrak{X}(\mathcal{W})\) such that \(\mathcal{L}_X \omega = \omega\).

**Proposition**

Let \(X\) be a Liouville vector field transverse to \(\Sigma\). Then \((\Sigma, \alpha = \iota_X \omega)\) is a contact manifold. If \(\Sigma = H^{-1}(c)\), then \(R_\alpha \cong X_H \big|_{H=c}\).

**Conjecture (Weinstein conjecture)**

Let \((M, \alpha)\) closed contact manifold. Then \(R_\alpha\) admits periodic orbits.
Contact Geometry of the RC3BP

- For \( c < H(L_1) \), \( \Sigma_c = H^{-1}(c) \) has 3 connected components: \( \Sigma^E_c \) (the satellite stays close to the earth), \( \Sigma^M_c \) (to the moon), or it is far away.

**Proposition (Albers–Frauenfelder–Koert–Paternain)**

For \( c < H(L_1) \), \( X = (q - q_E) \frac{\partial}{\partial q} \) is transverse to \( \Sigma^E_c \).

Hence \((\Sigma^E_c, \iota_X \omega)\) is contact.
But Weinstein conjecture does not apply because of non-compactness (collision!)
Moser regularization of the restricted 3-body problem

- Via Moser’s regularization $\Sigma_c^E$ can be compactified to $\overline{\Sigma}_c^E \cong \mathbb{R}P(3)$.
- The Liouville vector field $X = (q - q_E) \frac{\partial}{\partial q}$ extends to the regularization.
- Hence $\overline{\Sigma}_c^E$ is contact.

**Theorem (Albers–Frauenfelder–Koert–Paternain)**

For any value $c < H(L_1)$, the regularized RPC3BP has a closed orbit with energy $c$. 
But...

- Where are those periodic orbits?
- Maybe on the collision set?
- Keep track of the singularities in the geometric structure?
- \(b^m\)-symplectic and \(b^m\)-contact geometry!
$b^m$-stuff
Consider a hypersurface $Z = f^{-1}(0)$ of $M$.

$b\mathfrak{X}(M) = \{ \text{v.f. tangent to } Z \} = \{ f \frac{\partial}{\partial f}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}} \}$

**Serre–Swan:** There exists a bundle $bTM$ such that $\Gamma(bTM) = b\mathfrak{X}(M)$.

The dual: $bT^*M$ and forms: $b\Omega^k(M) = \Gamma(\wedge^k(bT^*M))$. 
\( \omega \in b\Omega^k(M) \) can be decomposed

\[
\omega = \alpha \wedge \frac{df}{f} + \beta \quad \text{where} \quad \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M).
\]
\[ \omega \in b\Omega^k(M) \text{ can be decomposed} \]

\[ \omega = \alpha \wedge \frac{df}{f} + \beta \text{ where } \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M). \]

Extension of the exterior derivative by defining

\[ d(\alpha \wedge \frac{df}{f} + \beta) := d\alpha \wedge \frac{df}{f} + d\beta. \]
\[ \omega \in b\Omega^k(M) \] can be decomposed

\[ \omega = \alpha \wedge \frac{df}{f} + \beta \text{ where } \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M). \]

Extension of the exterior derivative by defining

\[ d(\alpha \wedge \frac{df}{f} + \beta) := d\alpha \wedge \frac{df}{f} + d\beta. \]

Advantage: symplectic techniques work (often): Moser’s path method, Darboux theorem,...
A $b$-symplectic form on $W^{2n}$ is $\omega \in b\Omega^2(W)$ such that
- $d\omega = 0$,
- $\omega$ is non-degenerate.

A manifold $(M^{2n+1}, \alpha)$ where $\alpha \in b\Omega^1(M)$ is $b$-contact if $\alpha \wedge (d\alpha)^n \neq 0$. 
Proposition

Let \((W, Z, \omega)\) be a \(b\)\(^m\)-symplectic manifold and \(X \in b\)\(^m\)\(X(W)\) such that \(L_X \omega = \omega\) and \(X/\Sigma\). Then \((\Sigma, \iota_X \omega)\) is \(b\)\(^m\)-contact with critical set \(\tilde{Z} = Z \cap \Sigma\).
Proposition

Let \((W, Z, \omega)\) be a \(b^m\)-symplectic manifold and \(X \in {^b^m}\mathfrak{X}(W)\) such that \(\mathcal{L}_X \omega = \omega\) and \(X \pitchfork \Sigma\). Then \((\Sigma, \iota_X \omega)\) is \(b^m\)-contact with critical set \(\tilde{Z} = Z \cap \Sigma\).
Jacobi

Poisson

$b$-Symplectic

Symplectic

$b$-Contact

Contact

Cédric Oms (UPC)

$b^m$-Contact Geometry: Periodic Orbits and...
Dynamics on $b^m$-contact manifolds
The Reeb vector field $R_\alpha$ is defined by the equations

\[
\begin{aligned}
\iota_{R_\alpha} d\alpha &= 0 \\
\iota_{R_\alpha} \alpha &= 1.
\end{aligned}
\]

**Proposition**

Let $(M, \alpha = u \frac{dz}{z} + \beta)$ be a $b^m$-contact manifold of dimension 3. Then the restriction on $\tilde{Z}$ of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian with respect to $\Theta$ with Hamiltonian function $u$, i.e. $\iota_{R_\Theta} \Theta = du$.

Contrast to Daniel’s talk last week!
Proposition

Let \((M, \alpha)\) be a 3-dimensional \(b^m\)-contact manifold and assume the critical hypersurface \(Z\) to be closed. Then there exists infinitely many periodic Reeb orbits on \(Z\).

Proof.

1. \(\alpha = u \frac{dz}{z} + \beta\)
2. \(u\) is non-constant on \(Z\)
3. \(R_\alpha\) is Hamiltonian on \(Z\) for \(-u\),
4. \(u^{-1}(p)\) where \(p\) regular is a circle,
5. \(R_\alpha\) periodic on \(u^{-1}(p)\).
No periodic orbits away from $Z$

There are compact $b^m$-contact manifolds $(M, Z)$ of any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits on $M \setminus Z$.

**Example**

- $S^3 \subset (\mathbb{R}^4, \omega)$
- $X = \frac{1}{2} x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + \frac{1}{2} (x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2})$ Liouville v.f.
- $R_\alpha = 2x_1^2 \frac{\partial}{\partial x_1} - x_1 y_1 \frac{\partial}{\partial y_1} + 2x_2 \frac{\partial}{\partial y_2} - 2y_2 \frac{\partial}{\partial x_2}$
- On $Z = S^2$: rotation,
- Away from $Z$, no periodic orbits.
Periodic orbits away from $Z$?

**Definition**

$(M^3, \xi = \ker \alpha)$ is overtwisted if there exists $D^2$ s.t. $TD \cap \xi$ defines a 1-dimensional foliation given by

A contact manifold that is not overtwisted is called tight.
Theorem (Hofer ’93)

Let \((M^3, \alpha)\) a closed OT contact manifold. Then there exists a periodic orbit.

Not true for open OT manifolds!

Definition

A \(b^m\)-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface \(Z\).

Definition

A \(b^m\)-contact form \(\alpha\) is \(\mathbb{R}^+\)-invariant around the critical set if \(\alpha = udz + \beta\), where \(u \in C^\infty(Z)\) and \(\beta \in \Omega^1(Z)\).
Theorem

Let \((M, \alpha)\) be a closed \(b^m\)-contact manifold with critical set \(Z\). Assume there exists an overtwisted disk in \(M \setminus Z\) and assume that \(\alpha\) is \(\mathbb{R}^+\)-invariant in a tubular neighbourhood around \(Z\). Then there exists

1. a periodic Reeb orbit in \(M \setminus Z\) or
2. a family of periodic Reeb orbits approaching the critical set \(Z\).

Furthermore, the periodic orbits are contractible loops in the symplectization.

The proof is an adaptation of Hofer’s technique.
Consequence of more general theorem:

**Theorem**

Let $(M^3, \alpha)$ be a $\mathbb{R}^+$-invariant contact manifold that is OT away from the $\mathbb{R}^+$-invariant part. Then there exists a 1-parametric family of periodic Reeb orbits in the $\mathbb{R}^+$-invariant part of $M$ or a periodic Reeb orbit away from the $\mathbb{R}^+$-invariant part.

Mantra: Non-compactness $+ \mathbb{R}^+$-invariance $= \text{compactness}$

Question: Applications of this theorem?
Back to the motivating example
Contact geometry of RPC3BP revisited

In rotating coordinates: \( H(q, p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q_qE|} + \frac{\mu}{|q_qM|} + p_1 q_2 - p_2 q_1 \)

**Lemma**

The vector field \( Y = p \frac{\partial}{\partial p} \) is a Liouville vector field and is transverse to \( \Sigma_c \) for \( c > 0 \).

- Symplectic polar coordinates: \( (r, \alpha, P_r, P_\alpha) \).
- McGehee change of coordinates: \( r = \frac{2}{x^2} \).

\( b^3 \)-symplectic form: \( -4 \frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha \).

Is \( \Sigma_c \) \( b^3 \)-contact after McGehee? Can we apply the results on periodic orbits?
$b^3$-contact form in the RPC3BP

**Theorem**

*After the McGehee change, the Liouville vector field $Y = p \frac{\partial}{\partial p}$ is a $b^3$-vector field that is everywhere transverse to $\Sigma_c$ for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are $b^3$-contact manifolds.*

1. The critical set is a disjoint union of two cylinders.
$b^3$-contact form in the RPC3BP

Theorem

After the McGehee change, the Liouville vector field $\mathbf{Y} = p \frac{\partial}{\partial p}$ is a $b^3$-vector field that is everywhere transverse to $\Sigma_c$ for $c > 0$ and the level-sets $(\Sigma_c, \iota_{\mathbf{Y}} \omega)$ for $c > 0$ are $b^3$-contact manifolds.

1. The critical set is a disjoint union of two cylinders. 😊
After the McGehee change, the Liouville vector field $Y = p \frac{\partial}{\partial p}$ is a $b^3$-vector field that is everywhere transverse to $\Sigma_c$ for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are $b^3$-contact manifolds.

1. The critical set is a disjoint union of two cylinders. 😊
2. The Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.
Theorem

After the McGehee change, the Liouville vector field $Y = p \frac{\partial}{\partial p}$ is a $b^3$-vector field that is everywhere transverse to $\Sigma_c$ for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are $b^3$-contact manifolds.

1. The critical set is a disjoint union of two cylinders.

2. The Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.
Proof.

- $Y$ transverse at the critical set?
- On critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H)|_{H=c} = P_r^2 - P_\alpha = \frac{1}{2}P_r^2 + c > 0$;
- $b^3$-contact form $\alpha = (P_r \frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha)|x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;
- $R_\alpha|_Z = X_{P_r}$;
- Cylinder is foliated by periodic orbits.
Open questions

1. Can those periodic orbits be continued away from the critical set?
2. What about p.o. at $\infty$ for $\Sigma_c^{unbounded}$ for $c < H(L_1)$?
3. We are lucky here: $\dim Z = 2$. Results for p.o. for $(M, \alpha)$ higher dimension?
$b^m$-geometry wins!
Thanks for listening